

Heisenberg algebra and a graphical calculus

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Abstract

A new calculus of planar diagrams involving diagrammatics for biadjoint functors and degenerate affine Hecke algebras is introduced. The calculus leads to an additive monoidal category whose Grothendieck ring contains an integral form of the Heisenberg algebra in infinitely many variables. We construct bases of vector spaces of morphisms between products of generating objects in this category.

1 Introduction

In this paper we propose a graphical calculus for a categorification of the Heisenberg algebra. The one-variable Heisenberg algebra has generators p, q , one defining relation $pq - qp = 1$, and appears as the algebra of operators in the quantization of the harmonic oscillator. A fundamental role in the quantum field theory is played by its infinitely-generated analogue, the algebra with generators p_i, q_i , for i in some infinite set I , and relations

$$p_i q_j = q_j p_i + \delta_{i,j} 1, \quad p_i p_j = p_j p_i, \quad q_i q_j = q_j q_i. \quad (1)$$

In Section 2 we define a strict monoidal category \mathcal{H}' with two generating objects Q_+ and Q_- , and morphisms between tensor products of these objects given by linear combinations of certain planar diagrams modulo local relations. The category is \mathbb{k} -linear over a ground commutative ring \mathbb{k} , and we specialize \mathbb{k} to a field of characteristic 0. The endomorphism rings of tensor powers $Q_+^{\otimes n}$ and $Q_-^{\otimes n}$ contain the group algebra $\mathbb{k}[S_n]$ of the symmetric group. The symmetrization and antisymmetrization idempotents in $\mathbb{k}[S_n]$ produce objects in the Karoubi envelope \mathcal{H} of \mathcal{H}' . These objects can be viewed as symmetric and exterior powers of the generating objects Q_+ and Q_- . Consequently, we denote them by

$$S_+^n := S^n(Q_+), \quad \Lambda_+^n := \Lambda^n(Q_+), \quad S_-^n := S^n(Q_-), \quad \Lambda_-^n := \Lambda^n(Q_-), \quad (2)$$

and call them the symmetric and exterior powers of Q_+ and Q_- . When $n = 0$,

$$S_+^0 \cong S_-^0 \cong \Lambda_+^0 \cong \Lambda_-^0 \cong \mathbf{1},$$

where $\mathbf{1}$ is the identity object of the monoidal category \mathcal{H} , $\mathbf{1} \otimes M = M$ for any M . We also set

$$S_+^n = S_-^n = \Lambda_+^n = \Lambda_-^n = 0 \quad \text{if } n < 0.$$

Proposition 1 *There are canonical isomorphisms in \mathcal{H}*

$$\begin{aligned} S_-^n \otimes \Lambda_+^m &\cong (\Lambda_+^m \otimes S_-^n) \oplus (\Lambda_+^{m-1} \otimes S_-^{n-1}), \\ S_-^n \otimes S_-^m &\cong S_-^m \otimes S_-^n, \\ \Lambda_+^n \otimes \Lambda_+^m &\cong \Lambda_+^m \otimes \Lambda_+^n. \end{aligned}$$

These isomorphisms are constructed in Section 2.2. Since \mathcal{H} is monoidal, its Grothendieck group $K_0(\mathcal{H})$ is a ring. It has generators $[M]$ over objects M of \mathcal{H} and relations $[M_1] = [M_2] + [M_3]$ whenever $M_1 \cong M_2 \oplus M_3$. The multiplication is defined by $[M_1][M_2] := [M_1 \otimes M_2]$.

Corollary 1 *The following equalities hold in $K_0(\mathcal{H})$:*

$$\begin{aligned} [S_-^n][\Lambda_+^m] &= [\Lambda_+^m][S_-^n] + [\Lambda_+^{m-1}][S_-^{n-1}], \\ [S_-^n][S_-^m] &= [S_-^m][S_-^n], \\ [\Lambda_+^n][\Lambda_+^m] &= [\Lambda_+^m][\Lambda_+^n]. \end{aligned}$$

Let $H_{\mathbb{Z}}$ be the unital ring with generators $a_n, b_n, n \geq 1$ and defining relations

$$a_n b_m = b_m a_n + b_{m-1} a_{n-1}, \quad (3)$$

$$a_n a_m = a_m a_n, \quad (4)$$

$$b_n b_m = b_m b_n. \quad (5)$$

We simply rewrote relations in Corollary 1 using a_n in place of $[S_-^n]$ and b_m instead of $[\Lambda_+^m]$. Also set $a_0 = b_0 = 1$, $a_n = b_n = 0$ for $n < 0$, and require that the above relations hold for any $n, m \in \mathbb{Z}$. Any product of a 's and b 's can be converted into a linear combination with nonnegative integer coefficients of monomials in b 's times monomials in a 's,

$$b_{m_1} b_{m_2} \dots b_{m_k} a_{n_1} a_{n_2} \dots a_{n_r} \quad (6)$$

with $1 \leq m_1 \leq m_2 \leq \dots \leq m_k$, $1 \leq n_1 \leq n_2 \leq \dots \leq n_r$. The Bergman diamond lemma [4] tells us that this set of elements is a basis of $H_{\mathbb{Z}}$ viewed as a free abelian group. Let $H = H_{\mathbb{Z}} \otimes \mathbb{C}$ be the \mathbb{C} -algebra with the same generators and relations as $H_{\mathbb{Z}}$.

Forming generating functions

$$A(t) = 1 + a_1 t + a_2 t^2 + \dots, \quad B(u) = 1 + b_1 u + b_2 u^2 + \dots,$$

we can rewrite relations (3) as

$$A(t)B(u) = B(u)A(t)(1 + tu).$$

Let

$$\tilde{A}(t) = 1 + tA'(-t)A(-t), \quad \tilde{A}(t) = 1 + \tilde{a}_1 t + \tilde{a}_2 t^2 + \dots$$

It is easy to check that $\tilde{a}_1, \tilde{a}_2, \dots$ generate the same subalgebra of H as a_1, a_2, \dots , and that

$$\tilde{A}(t)B(u) = B(u)\tilde{A}(t) + \frac{tu}{1-tu}.$$

Coefficients of this equation give us relations (7) below

$$\tilde{a}_n b_m = b_m \tilde{a}_n + \delta_{n,m} 1, \quad (7)$$

$$\tilde{a}_n \tilde{a}_m = \tilde{a}_m \tilde{a}_n, \quad (8)$$

$$b_n b_m = b_m b_n. \quad (9)$$

Algebra H is isomorphic to the algebra generated by \tilde{a}_n 's, b_m 's, $n, m > 0$, with defining relations (7) - (9). This allows us to identify H with the Heisenberg algebra and $H_{\mathbb{Z}}$ with its integral form.

Corollary 1 gives a ring homomorphism

$$\gamma : H_{\mathbb{Z}} \longrightarrow K_0(\mathcal{H}) \quad (10)$$

that takes a_n to $[S_-^n]$ and b_n to $[\Lambda_+^n]$.

Theorem 1 *Map γ is injective.*

This theorem is proved in Section 3.3.

Conjecture 1 *Map γ is an isomorphism.*

If true, this conjecture would allow us to view the additive monoidal category \mathcal{H} as a categorification of the integral form $H_{\mathbb{Z}}$ of the Heisenberg algebra.

The degenerate affine Hecke algebra, which we call degenerate AHA following a suggestion of Etingof, was introduced by Drinfeld [14] in the $GL(n)$ case and by Lusztig [28] in the general case. Cherednik [9] classified finite-dimensional irreducible representations of degenerate AHA; its centralizing properties were studied by him and Olshanski in [10, 37]. We denote by DH_n the degenerate AHA for $GL(n)$, over the base field \mathbb{k} . Under the canonical homomorphism [9, 14] from DH_n to the group algebra $\mathbb{k}[S_n]$ of the symmetric group polynomial generators of DH_n go to the Jucys-Murphy elements. Okounkov and Vershik [35], [36] presented a detailed derivation of the basic representation theory of the symmetric group via these elements, see also [25, Chapter 2], [8, 13]. For some other uses of Jucys-Murphy's elements and degenerate AHA we refer the reader to [20, 30, 34, 41].

We will prove in Section 4 that the ring of endomorphisms of the object $Q_+^{\otimes n}$ in our category is isomorphic to the tensor product of DH_n and the polynomial algebra in infinitely many variables. Thus, the degenerate AHA for $GL(n)$ emerges naturally in our approach as part of a larger structure. Polynomial generators of DH_n acquire graphical interpretation in our calculus as right-twisted curls on strands. We also describe a basis, given diagrammatically, of vector spaces of morphisms between arbitrary tensor products of generators Q_+ and Q_- of \mathcal{H}' .

To prove our results, we construct a family of functors from \mathcal{H}' to the category \mathcal{S}' whose objects are compositions of induction and restriction functors between group algebras $\mathbb{k}[S_n]$ of the symmetric group, and morphisms are natural transformations between these functors. The image under these functors of the endomorphism of Q_+ given by the right curl diagram is the Jucys-Murphy element. The image of the counterclockwise circle diagram with k right curls is the k -th moment of the Jucys-Murphy element. Products of these moments were investigated in [20, 34, 39, 41] in relation to the asymptotic representation theory of the symmetric group and free probability. It also appears that our construction should be related to the circle of ideas considered by Guionnet, Jones and Shlyakhtenko [19] that intertwine planar algebras and free probability. In addition, one would hope for a relation between our calculus and the geometrization of the Heisenberg algebra via Hilbert schemes by Nakajima [32], [33] and Grojnowski [18], and for a possible link with Frenkel, Jing and Wang [16].

We discovered monoidal category \mathcal{H}' by considering compositions of induction and restriction functors for standard inclusions of symmetric group algebras $\mathbb{k}[S_n] \subset \mathbb{k}[S_{n+1}]$.

Induction and restriction functors for inclusions of finite groups are biadjoint, and biadjointness natural transformations can be depicted via cap and cup planar diagrams. Furthermore, the composition of two induction functors admits a natural endotransformation given by the right multiplication by the transposition $(n+1, n+2)$, an endomorphism of $\mathbb{k}[S_{n+2}]$ viewed as left $\mathbb{k}[S_{n+2}]$ -module and right $\mathbb{k}[S_n]$ -module. We denote this natural transformation by the crossing of two upward-oriented strands. Relations on compositions of the crossing, cup, and cap transformations that hold for all n (universal relations) are given by equations (11)-(13). These relations together with isotopies of diagrams are exactly the defining relations for the additive monoidal category \mathcal{H}' .

Cautis and Licata [7] introduced graded relatives of \mathcal{H}' and \mathcal{H} associated to finite subgroups of $SU(2)$, identified their Grothendieck rings with certain quantized Heisenberg algebras, and constructed an action of their categories on derived categories of coherent sheaves on Hilbert schemes of points on the ALE spaces. Hom spaces in Cautis-Licata monoidal categories carry a natural grading (absent in our case) with finite-dimensional homogeneous terms and vanishing negative degree homs on certain objects, leading to a proof that their analogue of the map γ is an isomorphism.

By themselves, Heisenberg algebras are rather simple constructs. Their value is in the structures that quantum field theory builds on top of them, for instance, the structures of vertex operator algebras. The problem posed by Igor Frenkel [15] to categorify just the simplest vertex operator algebra remains wide open - perhaps our paper will serve as a small step towards this goal.

Acknowledgments: On a number of occasions Igor Frenkel emphasized to the author the importance of categorifying various structures related to the symmetric functions and Heisenberg algebras. More recently, Tony Licata asked directly if there exists a categorification of the Heisenberg algebra, and this paper was inspired by his question. The author would like to acknowledge partial support by the National Science Foundation via grants DMS-0706924 and DMS-0739392.

2 A new graphical calculus

2.1 Local moves and their consequences

Fix a commutative ring \mathbb{k} and consider the following additive \mathbb{k} -linear monoidal category \mathcal{H}' generated by two objects Q_+ and Q_- . An object of \mathcal{H}' is a finite direct sum of tensor products $Q_{\epsilon_1} \otimes \cdots \otimes Q_{\epsilon_m}$, denoted Q_ϵ , where $\epsilon = \epsilon_1 \dots \epsilon_m$ are finite sequences of signs. Thus, $Q_{\epsilon\epsilon'} \cong Q_\epsilon \otimes Q_{\epsilon'}$ for sequences ϵ, ϵ' and their concatenation $\epsilon\epsilon'$. The unit object corresponds to the empty sequence: $\mathbf{1} = Q_\emptyset$.

The space of homomorphisms $\text{Hom}_{\mathcal{H}'}(Q_\epsilon, Q_{\epsilon'})$ for sequences ϵ, ϵ' is the \mathbb{k} -module generated by suitable planar diagrams, modulo local relations. The diagrams consist of oriented compact one-manifolds immersed into the plane strip $\mathbb{R} \times [0, 1]$, modulo rel

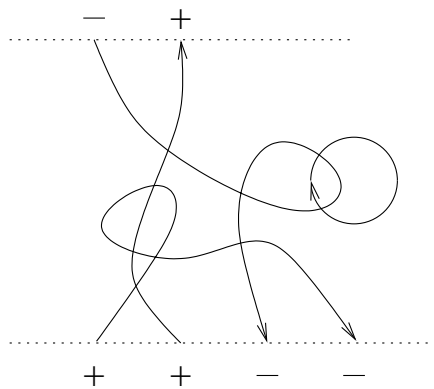
boundary isotopies. The relations are

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \begin{array}{c} \uparrow \\ \downarrow \end{array} \quad \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \begin{array}{c} \downarrow \\ \uparrow \end{array} - \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \quad (11)$$

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \begin{array}{c} \uparrow \\ \uparrow \end{array} \quad \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \quad (12)$$

$$\begin{array}{ccc}
\text{Diagram 1} & = & \mathbf{1} \\
\text{Diagram 2} & = & \mathbf{0}
\end{array}
\tag{13}$$

We require that the endpoints of the one-manifold are located at $\{1, \dots, m\} \times \{0\}$ and $\{1, \dots, k\} \times \{1\}$ and call these the lower and upper endpoints, respectively, where m and k are lengths of sequences ϵ and ϵ' . Moreover, orientation of the one-manifold at the endpoints must match the signs in the sequences ϵ and ϵ' . For instance, the diagram



is a morphism from Q_{++--} to Q_{--} . A diagram without endpoints is an endomorphism of $\mathbf{1}$. Composition of morphisms is given by concatenating the diagrams. The sequence of n pluses is denoted $+$ ^{n} , the sequence of n minuses $-$ ^{n} .

We have the Heisenberg relation

$$Q_{-+} \cong Q_{+-} \oplus \mathbf{1}.$$

This isomorphism is canonical and comes from the maps between these objects encoded in the following diagram

$$(14)$$

The four arrows are given by four morphisms, two of which are crossings and two are U-turns. The condition that these maps describe a decomposition of Q_{-+} as the direct sum of objects Q_{-+} and $\mathbf{1}$ is equivalent to relations (11) and (13), modulo the condition that an isotopy of a diagram does not change the morphism. The latter condition is equivalent to the biadjointness of functors of tensoring with Q_+ and Q_- , with the biadjointness transformations given by the four U -turns



(see Section 3.1 for details).

Moving the lower endpoints of a diagram up via a multiple cups diagram leads to canonical isomorphisms

$$\mathrm{Hom}_{\mathcal{H}'}(Q_\epsilon, Q_{\epsilon'}) \cong \mathrm{Hom}_{\mathcal{H}'}(\mathbf{1}, Q_{\bar{\epsilon}\epsilon'}) \cong \mathrm{Hom}_{\mathcal{H}'}(\mathbf{1}, Q_{\epsilon'\bar{\epsilon}}), \quad (15)$$

related to the biadjointness of tensoring with Q_ϵ and $Q_{\bar{\epsilon}}$. Here $\bar{\epsilon}$ is the sequence ϵ with the order and all signs reversed. Biadjointness natural transformations satisfy the cyclicity condition [1, 2, 11, 27], which follows at once from the definition of \mathcal{H}' .

The two relations in (11) allow simplification of a double crossing for oppositely oriented intervals. The first relation in (13) says that a counterclockwise oriented circle equals one. Thus, an innermost counterclockwise circle can be erased from the diagram without changing the value of the diagram viewed as an element of the hom space between the functors.

There are two possible types of curls on strands: a left curl  and a right curl .

The second relation in (13) says that a diagram that contains a left curl subdiagram is zero.

Defining local relations in \mathcal{H}' imply the following relations

$$\begin{array}{c} \text{circle with arrow up} \\ \hline \end{array} = \begin{array}{c} \text{vertical line with arrow up} \\ \hline \end{array} + \begin{array}{c} \text{circle with arrow down} \\ \hline \end{array}, \quad \text{crossing with arrows} = \text{crossing with arrows}.$$

The second relation, jointly with the original ones, implies that the triple intersection move holds for any orientation of the 3 strands:

$$\text{triple intersection} = \text{triple intersection}$$

Furthermore, right curls can be moved across intersection points, modulo simpler diagrams:

$$\begin{array}{c} \text{crossing with curl on top-left} \\ \hline \end{array} = \begin{array}{c} \text{crossing with curl on top-right} \\ \hline \end{array} + \begin{array}{c} \text{crossing with curl on bottom-left} \\ \hline \end{array} + \begin{array}{c} \text{crossing with curl on bottom-right} \\ \hline \end{array}$$

$$\begin{array}{c} \text{crossing with curl on top-left} \\ \hline \end{array} = \begin{array}{c} \text{crossing with curl on top-right} \\ \hline \end{array} + \begin{array}{c} \text{crossing with curl on bottom-left} \\ \hline \end{array} + \begin{array}{c} \text{crossing with curl on bottom-right} \\ \hline \end{array}$$

It will be convenient to denote a right curl by a dot on a strand, and k -th power of a right curl by a dot with k next to it:

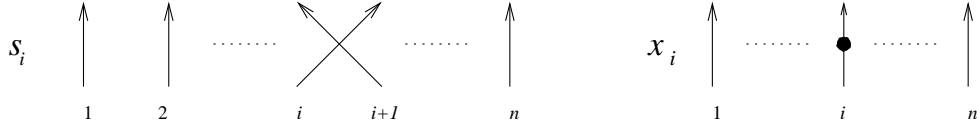
$$\begin{array}{c} \text{dot on strand} \\ \hline \end{array} := \text{right curl}, \quad k \begin{array}{c} \text{dot on strand} \\ \hline \end{array} := \begin{array}{c} \text{k dots} \\ \hline \end{array}$$

The above relations can be rewritten as

$$\begin{array}{c} \text{crossing with dot on top-left} \\ \hline \end{array} = \begin{array}{c} \text{crossing with dot on top-right} \\ \hline \end{array} + \begin{array}{c} \text{crossing with dot on bottom-left} \\ \hline \end{array} + \begin{array}{c} \text{crossing with dot on bottom-right} \\ \hline \end{array}$$

$$\begin{array}{c} \text{crossing with dot on top-left} \\ \hline \end{array} = \begin{array}{c} \text{crossing with dot on top-right} \\ \hline \end{array} + \begin{array}{c} \text{crossing with dot on bottom-left} \\ \hline \end{array} + \begin{array}{c} \text{crossing with dot on bottom-right} \\ \hline \end{array}$$

Together with the earlier ones, these relations show that there is a homomorphism from the degenerate affine Hecke algebra DH_n with coefficients in \mathbb{k} to the \mathbb{k} -algebra of endomorphisms of the object Q_{+^n} . The permutation generator s_i of DH_n goes to the permutation diagram of the i -th and $i + 1$ -st strands, and the polynomial generator x_i goes to the dot on the i -th strand:



Note that $\bigcirc = 0$. Indeed, the figure eight diagram, for any orientation, contains both left and right curls, and, therefore, equals to zero in our calculus.

A strand with k dots can be closed into either a clockwise-oriented or a counterclockwise-oriented circle with k dots. Denote these circles by c_k and \tilde{c}_k , respectively:

$$c_k := k \bigcirc, \quad \tilde{c}_k := k \bigcirc.$$

Counterclockwise circles can be expressed as linear combinations of products of clockwise circles. For the first few values of k , these are

$$\begin{aligned} \tilde{c}_0 &= 1, \\ \tilde{c}_1 &= 0, \\ \tilde{c}_2 &= c_0, \\ \tilde{c}_3 &= c_1, \\ \tilde{c}_4 &= c_2 + c_0^2, \\ \tilde{c}_5 &= c_3 + 2c_1c_0. \end{aligned}$$

These equations are obtained by expanding each dot into a left curl and then operating on the resulting diagram via the rules of the graphical calculus. A counterclockwise circle with one dot expands into the figure eight diagram, which is 0. For another example,

$$\tilde{c}_3 = \text{diagram} = \text{diagram} = \text{diagram} = c_1.$$

Proposition 2 For $k > 0$ we have

$$\tilde{c}_{k+1} = \sum_{a=0}^{k-1} \tilde{c}_a c_{k-1-a}. \quad (16)$$

Proof is the following computation:

$$\begin{aligned}
k+1 \text{ (circle with dot at left) } &= k \text{ (circle with dots at left and right) } = \text{ (figure-eight with dot at bottom) } \\
&= \text{ (figure-eight with dot at top) } + \sum_{a=0}^{k-1} \text{ (circle with dot at bottom labeled } a \text{) } \text{ (circle with dot at top labeled } k-1-a \text{) } \\
&= \sum_{a=0}^{k-1} \text{ (circle with dot at bottom labeled } a \text{) } \text{ (circle with dot at top labeled } k-1-a \text{) }
\end{aligned}$$

In the second equality, we converted a dot to a right curl, and, in the third equality, moved k dots through a crossing. The first term on the second line equals 0, since it contains a left curl. \square

Iterating this formula, one obtains an expression for \tilde{c}_k as a polynomial function of c_m , $m \leq k-2$. Vice versa, each c_m can be written as a polynomial in \tilde{c}_k , $k \leq m+2$. Let t be a formal variable and write

$$c(t) = \sum_{i=0}^{\infty} c_i t^i, \quad \tilde{c}(t) = \sum_{i=0}^{\infty} \tilde{c}_i t^i.$$

Formula (16) turns into $t^2 c(t) \tilde{c}(t) = \tilde{c}(t) - 1$, so that

$$\tilde{c}(t) = \frac{1}{1 - t^2 c(t)}.$$

The following identities, called *bubble moves* by analogy with [27], hold.

$$\begin{aligned}
\text{ (circle with dot at bottom labeled } k \text{) } \uparrow &= \uparrow \text{ (circle with dot at bottom labeled } k \text{) } + (k+1) \uparrow \text{ (dot) }^k - \sum_{b=0}^{k-2} (k-b-1) \uparrow \text{ (dot) }^{k-b-2} \text{ (circle with dot at bottom labeled } b \text{) } \\
\uparrow \text{ (circle with dot at bottom labeled } k \text{) } &= \text{ (circle with dot at bottom labeled } k \text{) } \uparrow - \sum_{b=0}^{k-2} (k-b-1) \text{ (circle with dot at bottom labeled } b \text{) } \uparrow \text{ (dot) }^{k-b-2}
\end{aligned}$$

A closed diagram D defines an endomorphism of the object $\mathbf{1}$ of \mathcal{H}' . Using the local moves, such diagram D can be converted into a linear combinations of crossingless diagrams that consist of nested dotted circles. Furthermore, bubble moves can be used to split apart nested circles. Lastly, convert counterclockwise circles into linear combinations of products of clockwise circles. Therefore, a closed diagram can be written as a linear combination of products of dotted clockwise circles. We see that the endomorphism algebra $\text{End}_{\mathcal{H}'}(\mathbf{1})$ is a quotient of the polynomial algebra $\Pi := \mathbb{k}[c_0, c_1, c_2, \dots]$ in countably many variables via the map

$$\psi_0 : \Pi = \mathbb{k}[c_0, c_1, c_2, \dots] \longrightarrow \text{End}_{\mathcal{H}'}(\mathbf{1}) \quad (17)$$

that takes c_k to the clockwise circle with k dots (we allowed ourselves the liberty of using c_k to denote both a formal variable and its image in the endomorphism algebra).

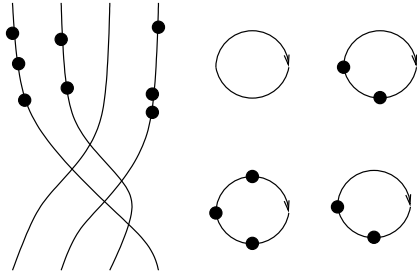
Proposition 3 *Map ψ_0 is an isomorphism.*

This proposition will be proved in Section 4.

The endomorphism algebra of Q_{+^m} is spanned by all diagrams that have m upper and m lower endpoints and such that at each endpoint the strand is oriented upward. A homomorphism from the degenerate affine Hecke algebra DH_m to $\text{End}_{\mathcal{H}'}(Q_{+^m})$ was described earlier. Placing a closed diagram to the right of a diagram representing an element of DH_m gives a homomorphism

$$\psi_m : \text{DH}_m \otimes \Pi \longrightarrow \text{End}_{\mathcal{H}'}(Q_{+^m}). \quad (18)$$

It is easy to see that ψ_m is surjective, by taking a diagram representing an element on the right hand side, and inductively simplifying it to a linear combination of diagrams that come from a standard basis of the left hand side. Namely, any diagram representing an endomorphism of Q_{+^m} is a combination of diagrams that consist of a permutation $\sigma \in S_m$, some number (possibly zero) of dots on each strand above the permutation diagram, and a monomial in dotted clockwise circles to the right:



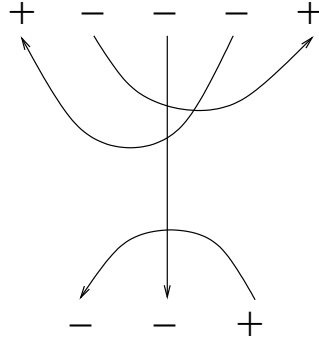
We write these basis elements as $x_1^{a_1} \dots x_m^{a_m} \cdot \sigma \cdot c_0^{b_0} c_1^{b_1} \dots c_k^{b_k}$. In the above example, the element is $x_1^3 x_2^2 x_4^3 \cdot (1324) \cdot c_0 c_2^2 c_3$.

Surjectivity of ψ_m can be strengthened to the following result.

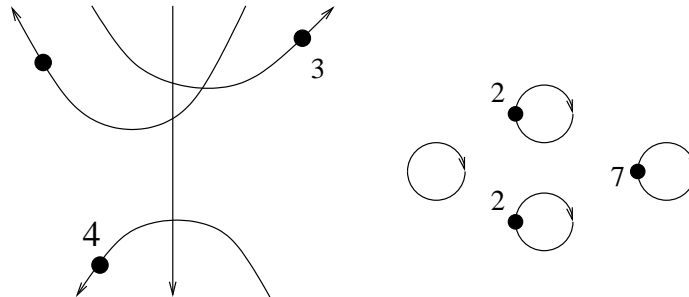
Proposition 4 ψ_m is an isomorphism.

Injectivity of ψ_m is proved in Section 4.

We now describe a spanning set in the \mathbb{k} -module $\text{Hom}_{\mathcal{H}'}(Q_\epsilon, Q_{\epsilon'})$ for any sequences ϵ, ϵ' . Let k be the total number of pluses in these two sequences. This hom space is nontrivial only if the total number of minuses in these two sequences is k as well, which we'll assume from now on to be the case. The spanning set, denoted $B(\epsilon, \epsilon')$, is obtained by forming all possible oriented matchings of these two points via k oriented segments in the plane strip $\mathbb{R} \times [0, 1]$. We assume that sequences ϵ and ϵ' are written at the bottom and top of the strip, the segments are embedded in the strip, and their orientations at the endpoints match corresponding elements of ϵ and ϵ' . Each two segments intersect at most once, and no triple intersections are allowed, see an example below for $\epsilon = - - +$ and $\epsilon' = + - - - +$.



Select an interval disjoint from intersections near the out endpoint of each interval and put any number (perhaps 0) of dots on it. In the rightmost region of the diagram, draw some number of clockwise oriented disjoint nonnested circles with no dots, some number of such circles with one dot, two dots, etc., with finitely-many circles in total. The resulting set of diagrams $B(\epsilon, \epsilon')$ is parametrized by $k!$ possible matchings of the $2k$ oriented endpoints, by a sequence of k nonnegative integers describing the number of dots on each interval, and by a finite sequence of nonnegative integers listing the number of clockwise oriented circles with no dots, one dot, and so on. An example of a diagram in $B(- - +, + - - - +)$ is depicted below.



In this picture, we put no dots, one dot, three dots, and four dots on the four arcs of the matching, added one bubble with no dots, two with two dots and one with five dots.

It is rather straightforward to check that $B(\epsilon, \epsilon')$ is a spanning set of the \mathbb{k} -vector space $\text{Hom}_{\mathcal{H}'}(Q_\epsilon, Q_{\epsilon'})$.

Proposition 5 *For any sign sequences ϵ, ϵ' the set $B(\epsilon, \epsilon')$ constitutes a basis of the \mathbb{k} -vector space $\text{Hom}_{\mathcal{H}'}(Q_\epsilon, Q_{\epsilon'})$.*

Thus, we claim that the set $B(\epsilon, \epsilon')$ is also linearly independent. This proposition holds as well for \mathbb{k} being any commutative ring rather than a field, with $B(\epsilon, \epsilon')$ being a basis of the free \mathbb{k} -module $\text{Hom}_{\mathcal{H}'}(Q_\epsilon, Q_{\epsilon'})$.

Notice that Proposition 4 is a special case of this proposition, for $\epsilon = \epsilon' = +^m$. Proposition 5 follows from proposition 4, functor isomorphisms $Q_{-+} \cong Q_{+-} \oplus \text{Id}$ and arguments similar to the ones in [24, Section 2.2]. First, canonical isomorphisms (15) take the set $B(\epsilon, \epsilon')$ to $B(\emptyset, \bar{\epsilon}\epsilon')$ and $B(\emptyset, \epsilon'\bar{\epsilon})$, respectively, and it is then enough to show that $B(\emptyset, \epsilon)$ is linearly independent for any sequence ϵ with k pluses and k minuses.

Proposition 4 implies linear independence for $k = 0, 1$, and for the sequence $+^k -^k$ for any k . Assume that $\epsilon = \epsilon_1 - + \epsilon_2$ for some sequences ϵ_1, ϵ_2 . Assume by induction on k and by induction on the lexicographic order among length $2k$ sequences that the sets $B(\emptyset, \epsilon_1\epsilon_2)$ and $B(\emptyset, \epsilon_1 + - \epsilon_2)$ are linearly independent in their respective hom spaces.

Two upper arrows in the diagram (14) lead to a canonical decomposition

$$Q_{\epsilon_1 - + \epsilon_2} \cong Q_{\epsilon_1 + - \epsilon_2} \oplus Q_{\epsilon_1 \epsilon_2}.$$

Under this isomorphism sets $B(\emptyset, \epsilon_1\epsilon_2)$ and $B(\emptyset, \epsilon_1 + - \epsilon_2)$ get mapped to two subsets of $\text{Hom}_{\mathcal{H}'}(\mathbf{1}, Q_{\epsilon_1 - + \epsilon_2})$. Denote by B the union of these two subsets. It is easy to see that linear independence of $B(\emptyset, \epsilon_1 - + \epsilon_2)$ is equivalent to linear independence of B , which we know by induction. Proposition 5 follows. \square

By the thickness of a diagram in $B(\epsilon, \epsilon')$ we call the number of arcs connecting lower and upper endpoints. The diagram depicted earlier has thickness one. For each k and ϵ , the subset of diagrams of thickness at most k is a 2-sided ideal in the endomorphism ring of Q_ϵ , since thickness cannot increase upon composition. For $\epsilon = +^n -^m$ and $k = n + m - 1$ we denote the corresponding ideal by $J_{n,m}$. It is spanned by diagrams with at least one arc connecting a pair of upper endpoints (and, necessarily, at least one arc connecting a pair of lower endpoints). It is easy to see that the quotient of the endomorphism ring of $Q_{+^n -^m}$ by this ideal is naturally isomorphic to the tensor product $DH_n \otimes DH_m^{op} \otimes \Pi$, and the short exact sequence

$$0 \longrightarrow J_{n,m} \longrightarrow \text{End}_{\mathcal{H}'}(Q_{+^n -^m}) \longrightarrow DH_n \otimes DH_m^{op} \otimes \Pi \longrightarrow 0 \quad (19)$$

admits a canonical splitting. Notice also that $DH_m^{op} \cong DH_m$.

We now list some obvious symmetries of \mathcal{H}' . The map that assigns $(-1)^{w(D)}D$ to a diagram D , where $w(D)$ is the number of crossings plus the number of dots of D , extends to an involutive autoequivalence ξ_1 of \mathcal{H}' . We have $\xi_1^2 = \text{Id}$ (equality and not just isomorphism). Autoequivalence ξ_1 exchanges S_+^n with Λ_+^n and S_-^n with Λ_-^n .

Denote by ξ_2 the symmetry of category \mathcal{H}' given on diagrams by reflecting about the x-axis and reversing orientation. This symmetry is an involutive monoidal contravariant autoequivalence of \mathcal{H}' .

Denote by ξ_3 the symmetry of category \mathcal{H}' given on diagrams by reflecting about the y-axis and reversing orientation. This symmetry is an involutive antimonoidal autoequivalence of \mathcal{H}' . Being antimonoidal means that it reverses the order of elements in the tensor product: $\xi_3(M \otimes N) = \xi_3(N) \otimes \xi_3(M)$.

Symmetries ξ_1, ξ_2, ξ_3 pairwise commute and generate an action of $(\mathbb{Z}/2)^3$.

2.2 Karoubi envelope and projectors

The two relations in (12) tell us that upward oriented crossings satisfy the symmetric group relations and give us a canonical homomorphism

$$\mathbb{k}[S_n] \longrightarrow \text{End}_{\mathcal{H}'}(Q_{+^n})$$

from the group algebra of the symmetric group to the endomorphism ring of the n -th tensor power of Q_+ . Turning the diagrams by 180 degrees, we obtain a canonical homomorphism

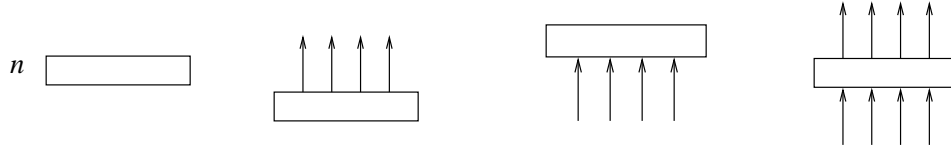
$$\mathbb{k}[S_n] \longrightarrow \text{End}_{\mathcal{H}'}(Q_{-^n}).$$

Assume that \mathbb{k} is a field of characteristic 0. Then we can use symmetrizers and antisymmetrizers, and, more generally, Young symmetrizers, to produce idempotents in $\text{End}_{\mathcal{H}'}(Q_{+^n})$. At this point it is convenient to introduce the Karoubi envelope of \mathcal{H}' , the category \mathcal{H} whose objects are pairs (P, e) , where P is an object of \mathcal{H}' and $e : P \longrightarrow P$ is an idempotent endomorphism, $e^2 = e$. Morphisms from (P, e) to (P', e') are maps $f : P \longrightarrow P'$ in \mathcal{H}' such that $e'fe = f$. It is immediate that \mathcal{H} is a \mathbb{k} -linear additive monoidal category.

To the complete symmetrizer

$$e(n) \in \mathbb{k}[S_n], \quad e(n) = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma$$

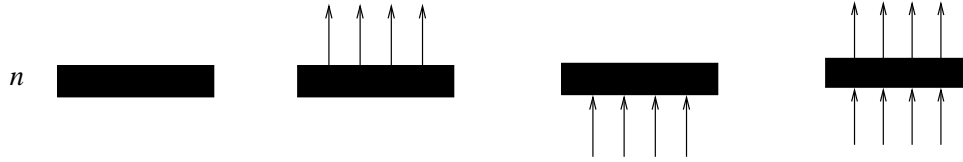
we assign the object $S_+^n := (Q_{+^n}, e(n))$ in \mathcal{H} . Following Cvitanović [12], which contains diagrammatics for Young symmetrizers and antisymmetrizers, we depict S_+^n as a white box labelled n . The inclusion morphism $S_+^n \longrightarrow Q_{+^n}$ is depicted by a white box with n upward oriented lines emanating from the top. The projection $Q_{+^n} \longrightarrow S_+^n$ is depicted by a white box with n upward oriented lines at the bottom. The composition $Q_{+^n} \longrightarrow S_+^n \longrightarrow Q_{+^n}$ is depicted likewise.



To the complete antisymmetrizer

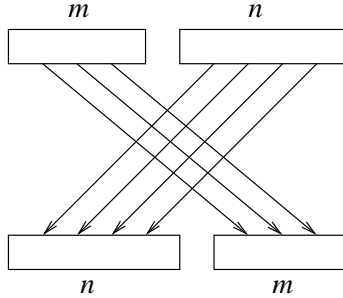
$$e'(n) \in \mathbb{k}[S_n], \quad e'(n) = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sign}(\sigma) \sigma$$

we assign the object $\Lambda_+^n := (Q_{+^n}, e'(n))$ in \mathcal{H} and depict it and related inclusions and projections to and from Q_{+^n} by black boxes with up arrows

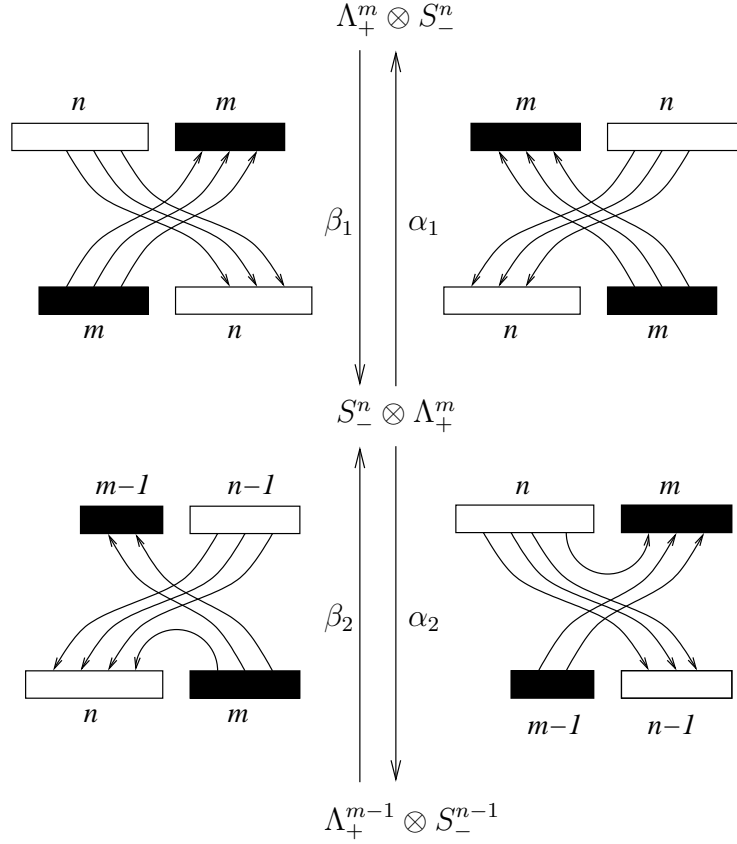


Define the objects $S_n^- := (Q_{-n}, e(n))$ and $\Lambda_n^- := (Q_{-n}, e'(n))$ as the subobjects of Q_{-n} associated to the symmetrizer $e(n)$ and the antisymmetrizer $e'(n)$ idempotents, respectively, under the canonical homomorphism $\mathbb{k}[S_n] \rightarrow \text{End}_{\mathcal{H}'}(Q_{-n})$. We draw S_n^- and Λ_n^- as white, respectively black, boxes, but with the lines at the boxes oriented downward.

We plan to develop the graphical calculus of these diagrams elsewhere. Part of the calculus that deals with the lines oriented only upwards (or only downwards) is the graphical calculus of symmetrizers and antisymmetrizers in the symmetric group, and can be found in [12]. The latter calculus implies the second and third isomorphisms from Proposition 1 in the introduction. For instance, the second isomorphism is realized by the diagram



The first family of isomorphisms in Proposition 1 is realized by the maps



A straightforward manipulation of diagrams shows that

$$\alpha_1 \beta_1 = \text{Id}, \quad \alpha_1 \beta_2 = 0, \quad \alpha_2 \beta_2 = \frac{1}{mn} \text{Id}, \quad \alpha_2 \beta_1 = 0.$$

Let $\beta'_2 = mn\beta_2$. Then the maps

$$\Lambda_+^m \otimes S_-^n \xrightleftharpoons[\beta_1]{\alpha_1} S_-^n \otimes \Lambda_+^m \xrightleftharpoons[\beta'_2]{\alpha_2} \Lambda_+^{m-1} \otimes S_-^{n-1}$$

satisfy

$$\alpha_1 \beta_1 = \text{Id}, \quad \alpha_1 \beta'_2 = 0, \quad \alpha_2 \beta'_2 = \text{Id}, \quad \alpha_2 \beta_1 = 0, \quad \beta_1 \alpha_1 + \beta'_2 \alpha_2 = \text{Id}.$$

The last equality follows from a direct diagrammatic manipulation as well. Thus, there is an isomorphism

$$S_-^n \otimes \Lambda_+^m \cong (\Lambda_+^m \otimes S_-^n) \oplus (\Lambda_+^{m-1} \otimes S_-^{n-1}),$$

concluding the proof of Proposition 1 and Corollary 1.

There is a natural bijection between partitions λ of n and (isomorphism classes of) irreducible representations of $\mathbb{k}[S_n]$. To each partition $\lambda = (\lambda_1, \dots, \lambda_k)$ with $|\lambda| =$

$\lambda_1 + \cdots + \lambda_k = n$ there corresponds the unique common irreducible summand L_λ of the representation induced from the trivial representation of parabolic subgroup $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_k} \subset S_n$ and the representation induced from the sign representation of parabolic subgroup $S_{\lambda^*} = S_{\lambda_1^*} \times \cdots \times S_{\lambda_k^*}$, where λ^* is the dual partition. Let $e_\lambda \in \mathbb{k}[S_n]$ be the Young idempotent, so that $e_\lambda^2 = e_\lambda$ and $L_\lambda \cong \mathbb{k}[S_n]e_\lambda$.

We denote by $Q_{+,\lambda} := (Q_{+^n}, e_\lambda)$ the object of \mathcal{H} which is the direct summand of Q_{+^n} corresponding to the idempotent e_λ , where we view the latter as an idempotent in the endomorphism ring via the standard homomorphism $\mathbb{k}[S_n] \rightarrow \text{End}_{\mathcal{H}}(Q_{+^n})$. Likewise, let $Q_{-,\lambda} := (Q_{-^n}, e_\lambda)$ be the corresponding direct summand of Q_{-^n} , where we view e_λ as an endomorphism of the latter object. In particular,

$$S_+^n = Q_{+,(n)}, \quad \Lambda_+^n = Q_{+,(1^n)}, \quad S_-^n = Q_{-,(n)}, \quad \Lambda_-^n = Q_{-,(1^n)}.$$

The Grothendieck ring $K_0(\mathcal{H})$ is an abelian group with generators—symbols $[M]$, over all objects M of \mathcal{H} and defining relations $[M_1] = [M_2] + [M_3]$ whenever $M_1 \cong M_2 \oplus M_3$. Monoidal structure on \mathcal{H} descends to an associative multiplication on $K_0(\mathcal{H})$, with $[1]$ being the identity for multiplication. Hence, $K_0(\mathcal{H})$ is an associative unital ring.

Recall the ring $H_{\mathbb{Z}}$ from the introduction. We can now define homomorphism $\gamma : H_{\mathbb{Z}} \rightarrow K_0(\mathcal{H})$ discussed there:

$$\gamma(a_n) = [Q_{-,(n)}] = [S_-^n], \quad \gamma(b_m) = [Q_{+,(1^m)}] = [\Lambda_+^m].$$

If we identify the subring of $H_{\mathbb{Z}}$ generated by the a_n 's with the ring of symmetric functions Sym so that a_n corresponds to n -th complete symmetric function h_n , then γ will take the Schur function associated to partition λ to $[Q_{+,\lambda}]$. This function is often denoted s_λ ; for us it is convenient to call it a_λ , so that $a_{(n)} = a_n$.

Similarly, we identify the subring generated by the b_m 's with Sym by taking b_m to the m -th elementary symmetric function e_m . Denote by b_λ the polynomial in b_m 's that corresponds to the Schur function s_λ under this identification. In particular, $b_{(1^m)} = b_m$. We have

$$\gamma(a_\lambda) = [Q_{-,\lambda}], \quad \gamma(b_\lambda) = [Q_{+,\lambda^*}].$$

Littlewood-Richardson coefficients $r_{\lambda,\mu}^\nu$ that appear in decompositions of the product of Schur functions

$$\begin{aligned} a_\lambda a_\mu &= \sum_{\nu} r_{\lambda,\mu}^\nu a_\nu, \\ b_\lambda b_\mu &= \sum_{\nu} r_{\lambda,\mu}^\nu b_\nu, \end{aligned}$$

also appear in the isomorphism formulas in \mathcal{H} :

$$\begin{aligned} Q_{+,\lambda} \otimes Q_{+,\mu} &\cong \bigoplus_{\nu} (Q_{+,\nu})^{r_{\lambda,\mu}^\nu}, \\ Q_{-,\lambda} \otimes Q_{-,\mu} &\cong \bigoplus_{\nu} (Q_{-,\nu})^{r_{\lambda,\mu}^\nu}. \end{aligned}$$

Descending to the Grothendieck ring, we have

$$\begin{aligned}[Q_{+,\lambda}][Q_{+,\mu}] &= \sum_{\nu} r_{\lambda,\mu}^{\nu} [Q_{+,\nu}], \\ [Q_{-,\lambda}][Q_{-,\mu}] &= \sum_{\nu} r_{\lambda,\mu}^{\nu} [Q_{-,\nu}].\end{aligned}$$

The ring $H_{\mathbb{Z}}$ has a basis $\{b_{\mu}a_{\lambda}\}_{\lambda,\mu}$ over all partitions λ, μ . Consequently, elements $[Q_{+,\mu}][Q_{-,\lambda}]$ over all λ, μ span the subring $\gamma(H_{\mathbb{Z}})$ of $K_0(\mathcal{H})$.

Remark: The symmetries ξ_1, ξ_2, ξ_3 of \mathcal{H}' extend to self-equivalences of category \mathcal{H} , also denoted ξ_1, ξ_2, ξ_3 . On objects $Q_{+,\lambda} \otimes Q_{-,\mu}$ they act as follows:

$$\begin{aligned}\xi_1(Q_{+,\mu} \otimes Q_{-,\lambda}) &= Q_{+,\mu}^* \otimes Q_{-,\lambda}^*, \\ \xi_2(Q_{+,\mu} \otimes Q_{-,\lambda}) &= Q_{+,\mu} \otimes Q_{-,\lambda}, \\ \xi_3(Q_{+,\mu} \otimes Q_{-,\lambda}) &= Q_{+,\lambda} \otimes Q_{-,\mu}.\end{aligned}$$

These self-equivalences induce involutions $[\xi_1], [\xi_2]$ and antiinvolution $[\xi_3]$ on $K_0(\mathcal{H})$. The involution of $H_{\mathbb{Z}}$ corresponding to $[\xi_2]$ is the identity. We don't know whether $[\xi_2]$ is the identity involution on the entire $K_0(\mathcal{H})$; this would follow from Conjecture 1.

3 Diagrammatics for induction and restriction functors

3.1 Biadjoint functors

Recall [29] that a functor $L : \mathcal{A} \rightarrow \mathcal{B}$ between categories \mathcal{A} and \mathcal{B} is left adjoint to a functor $R : \mathcal{B} \rightarrow \mathcal{A}$ whenever there are natural transformations

$$\alpha : LR \Rightarrow \text{Id}_{\mathcal{B}}, \quad \beta : \text{Id}_{\mathcal{A}} \Rightarrow RL \quad (20)$$

that satisfy the relations

$$(\alpha \circ \text{Id}_L)(\text{Id}_L \circ \beta) = \text{Id}_L, \quad (\text{Id}_R \circ \alpha)(\beta \circ \text{Id}_R) = \text{Id}_R. \quad (21)$$

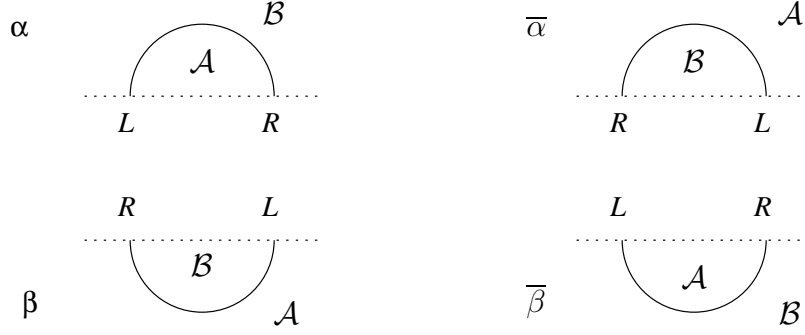
Assume that L is both left and right adjoint to R , and the second adjunction maps

$$\bar{\alpha} : RL \Rightarrow \text{Id}_{\mathcal{A}}, \quad \bar{\beta} : \text{Id}_{\mathcal{B}} \Rightarrow LR \quad (22)$$

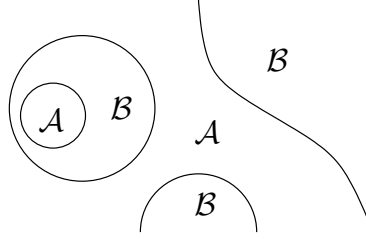
are fixed as well. They satisfy

$$(\bar{\alpha} \circ \text{Id}_R)(\text{Id}_R \circ \bar{\beta}) = \text{Id}_R, \quad (\text{Id}_L \circ \bar{\alpha})(\bar{\beta} \circ \text{Id}_L) = \text{Id}_L. \quad (23)$$

Out of $\alpha, \bar{\alpha}, \beta, \bar{\beta}$ one can construct more general natural transformations between compositions of functors L and R by placing the basic four transformation in various locations in the composition of functors, and then composing several such transformations. It's convenient to draw these compositions via planar diagrams, with transformations (20), (22) depicted as U-turns



General diagrams are built out of U-turns and vertical lines, the latter denoting identity natural transformations of R and L . For instance,



is the following natural transformation from RLR to R

$$(\bar{\alpha} \circ \text{Id})(\text{Id} \circ \alpha \circ \text{Id}^{\otimes 2})(\text{Id} \circ \bar{\beta} \circ \text{Id}^{\otimes 2})(\beta \circ \text{Id})\bar{\alpha}$$

The four biadjointness equations (21), (23), which can be drawn as

$$\begin{array}{cc} \begin{array}{c} \mathcal{B} \\ \text{wavy line} \\ \mathcal{A} \end{array} = \mathcal{B} \Big| \mathcal{A} & \begin{array}{c} \mathcal{B} \\ \text{wavy line} \\ \mathcal{A} \end{array} = \mathcal{A} \Big| \mathcal{B} \\ \begin{array}{c} \mathcal{A} \\ \text{wavy line} \\ \mathcal{B} \end{array} = \mathcal{A} \Big| \mathcal{B} & \begin{array}{c} \mathcal{A} \\ \text{wavy line} \\ \mathcal{B} \end{array} = \mathcal{B} \Big| \mathcal{A} \end{array}$$

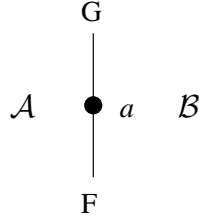
are equivalent to the condition that the lines and circles can be isotoped without changing the natural transformation associated to the diagram. This was observed in [11, 31].

The graphical calculus of biadjoints can be further enhanced. Assume given a collection of categories and a collection of functors between them such that each functor has a biadjoint, which is also in the collection, and the biadjointness transformations are fixed. Natural transformations generated by the biadjointness ones can be drawn

via diagrams on the plane strip $\mathbb{R} \times [0, 1]$, with lines and circles labelled by functors and regions labelled by categories, with arbitrary rel boundary isotopies allowed.

Furthermore, any element z in the center of a category \mathcal{A} (i.e. z is the endomorphism of the identity functor $\text{Id}_{\mathcal{A}}$) can be shown as freely floating in a region labelled \mathcal{A} . Two such central elements can freely move past each other.

Given two functors $F, G : \mathcal{A} \longrightarrow \mathcal{B}$ in the collection, a natural transformation $a : F \longrightarrow G$ can be depicted as a labelled dot on a line separating a segment labelled F from a segment labelled G .



When a is dragged through a U -turn, it can change in two possible ways, depending on the type of a U -turn, into natural transformations $a^*, {}^*a : G' \longrightarrow F'$, where F', G' are the biadjoints of F and G . If ${}^*a = a^*$, we say that the biadjointness data is *cyclic*. For more on the cyclicity condition see [1, 2, 11, 27].

Biadjoint functors appear throughout categorification and TQFTs, see discussions in [21, Section 5], [22] and references therein, also [6].

3.2 Induction and restriction between finite groups in pictures

We fix a commutative ring \mathbb{k} and denote by $\mathbb{k}G$ the group algebra of finite group G with coefficients in \mathbb{k} . We denote by (G) the group algebra $\mathbb{k}G$, viewed as a $(\mathbb{k}G, \mathbb{k}G)$ -bimodule. If H is a subgroup of G and we view $\mathbb{k}G$ as a $(\mathbb{k}G, \mathbb{k}H)$ -bimodule via the left action of $\mathbb{k}G$ and the right action of $\mathbb{k}H$, we denote it by (G_H) . When viewing $\mathbb{k}G$ as a $\mathbb{k}H$ -bimodule, denote it by $({}_H G_H)$, etc. Similar shortcut notation is adopted for tensor products of bimodules. For instance, $(G_H G)$ denotes $\mathbb{k}G$ -bimodule $\mathbb{k}G \otimes_{\mathbb{k}H} \mathbb{k}G$, while $({}_H G_H G)$ denotes the same space, but viewed as a $(\mathbb{k}H, \mathbb{k}G)$ -bimodule.

Start with the 2-category $BFin$ whose objects are finite groups G , morphisms from G to H are $(\mathbb{k}H, \mathbb{k}G)$ -bimodules, and 2-morphisms are bimodule homomorphisms. Consider the 2-subcategory $IRFin'$ of $BFin$ with the same objects as $BFin$, while morphisms are finite direct sums of tensor products of bimodules (G_H) and $({}_H G)$ corresponding to the induction and restriction functors between categories of H and G -modules. Thus, a 1-morphism from G to G' is a finite direct sum of bimodules isomorphic to

$$(G_n {}_{H_{n-1}} G_{n-1} {}_{H_{n-2}} \cdots {}_{H_2} G_2 {}_{H_1} G_1)$$

where $G' = G_n \supset H_{n-1} \subset G_{n-1} \supset \cdots \subset G_2 \supset H_1 \subset G_1 = G$ is a zigzag of inclusions between finite groups. The 2-morphisms in $IRFin'$ are bimodule homomorphisms.

Alternatively, we can think of 1-morphisms in $IRFin'$ as given by compositions of induction and restriction functors between categories of (left) G -modules, over finite groups G .

In this section, we develop basics of a graphical calculus for studying 2-morphisms in $IRFin'$. In general, given a unital inclusion of rings $B \subset A$, the induction functor $\text{Ind} : B\text{-mod} \rightarrow A\text{-mod}$ that takes M to $A \otimes_B M$ is left adjoint to the restriction functor. An inclusion $\iota : H \subset G$ of finite groups produces an inclusion $\mathbb{k}H \subset \mathbb{k}G$ of group algebras, with the induction functor

$$\text{Ind}_H^G : \mathbb{k}H\text{-mod} \rightarrow \mathbb{k}G\text{-mod}$$

being both left and right adjoint (i.e. biadjoint) to the restriction functor

$$\text{Res}_G^H : \mathbb{k}G\text{-mod} \rightarrow \mathbb{k}H\text{-mod}.$$

The biadjointness endomorphisms are given by the following four bimodule maps

- 1) : $(G_H G) \rightarrow (G), \quad x \otimes y \mapsto xy, \quad x, y \in (G),$
- 2) : $(H) \rightarrow ({}_H G_G G_H), \quad x \mapsto x \otimes 1 = 1 \otimes x, \quad x \in (H),$
- 3) : $({}_H G_G G_H) \cong ({}_H G_H) \rightarrow (H), \quad g \mapsto g \text{ if } g \in H, g \mapsto 0, \text{ if } g \in G \setminus H.$

We denote this projection map by $p_H : ({}_H G_H) \rightarrow (H), \quad p_H(g) = g \text{ if } g \in H \text{ and } p_H(g) = 0 \text{ if } g \in G \setminus H,$ extended by \mathbb{k} -linearity. Clearly, p_H is a map of $\mathbb{k}H$ -bimodules.

4) Let $G = \bigsqcup_{i=1}^m H g_i$ be a decomposition of G into left H -cosets, so that $m = [G : H]$ is the index of H in G . Notice that the element

$$\sum_{i=1}^m g_i^{-1} \otimes g_i \in (G_H G)$$

does not depend on the choice of coset representatives $\{g_i\}_{i=1}^m$ of H in G : if $g'_i = h_i g_i$ then

$$\sum_{i=1}^m g_i'^{-1} \otimes g'_i = \sum_{i=1}^m g_i^{-1} h_i^{-1} \otimes h_i g_i = \sum_{i=1}^m g_i^{-1} \otimes g_i,$$

since the tensor product is over $\mathbb{k}[H]$, and h_i can be moved through the tensor product sign. Define bimodule map

$$(G) \rightarrow (G_H G) \tag{24}$$

by the condition that

$$1 \mapsto \sum_{i=1}^m g_i^{-1} \otimes g_i,$$

so that

$$g \mapsto \sum_{i=1}^m g_i^{-1} \otimes g_i g = \sum_{i=1}^m g g_i^{-1} \otimes g_i.$$

The second equality, needed to insure that one does get a bimodule map, follows from the following computation: $g_i g = h_i g_{i'}$ for some $h_i \in H$ and $i' \in \{1, 2, \dots, m\}$. The assignment $i \mapsto i'$ is a bijection of $\{1, 2, \dots, m\}$. We have

$$\sum_{i=1}^m g_i^{-1} \otimes g_i g = \sum_{i=1}^m g_i^{-1} \otimes h_i g_{i'} = \sum_{i=1}^m g_i^{-1} h_i \otimes g_{i'} = \sum_{i'=1}^m g g_{i'}^{-1} \otimes g_{i'} = \sum_{i=1}^m g g_i^{-1} \otimes g_i.$$

Combining with the earlier remark, we see that (24) is a bimodule map which does not depend on the choices of H -coset representatives g_i .

We associate to these four bimodule maps the following four pictures

$$\begin{array}{cccc} \begin{array}{c} G \\ \curvearrowright \\ H \\ \curvearrowleft \\ G \end{array} & \begin{array}{c} \\ \curvearrowright \\ H \end{array} & \begin{array}{c} H \\ \curvearrowleft \\ G \\ \curvearrowright \\ H \end{array} & \begin{array}{c} \\ \curvearrowleft \\ G \end{array} \end{array}$$

Thus, $\begin{array}{c} G \\ \curvearrowright \\ H \\ \curvearrowleft \\ G \end{array}$, denoted α_H^G , is the map

$$(G_H G) \longrightarrow (G), \quad x \otimes y \longmapsto xy, \quad x, y \in (G).$$

$\begin{array}{c} \\ \curvearrowright \\ H \end{array}$, denoted β_H^G , is the map

$$(H) \longrightarrow ({}_H G_G G_H) \cong ({}_H G_H), \quad x \longmapsto x \otimes 1 = 1 \otimes x, \quad x \in (H).$$

$\begin{array}{c} H \\ \curvearrowleft \\ G \\ \curvearrowright \\ H \end{array}$, denoted $\overline{\alpha}_G^H$, is the map p_H described earlier,

$$({}_H G_G G_H) \cong ({}_H G_H) \longrightarrow (H), \quad x \longmapsto p_H(x), \quad x \in (G).$$

$\begin{array}{c} \\ \curvearrowleft \\ G \end{array}$, denoted $\overline{\beta}_G^H$, is the \mathbb{k} -linear map (24)

$$(G) \longrightarrow (G_H G), \quad g \longmapsto \sum_{i=1}^m g_i^{-1} \otimes g_i g, \quad g \in G.$$

Theorem 2 *These four bimodule maps turn induction and restriction functors Ind_H^G and Res_G^H into a cyclic biadjoint pair.*

Proof: First, we check that the adjointness equations (21) and (23) hold for these maps. The bimodule map $(G_H) \longrightarrow (G_H G_H) \longrightarrow (G_H)$ corresponding to the left hand side of the first equation in (21) is given by $g \longmapsto g \otimes 1 \longmapsto g1 = g$, hence the map is the identity:

$$\begin{array}{c} G \\ \curvearrowright \\ H \end{array} = \begin{array}{c} G \\ \uparrow \\ H \end{array}$$

The bimodule map $({}_H G) \longrightarrow ({}_H G {}_H G) \longrightarrow ({}_H G)$ for the left hand side of the second equation in (21) is given by $g \longmapsto 1 \otimes g \longmapsto 1g = g$, and the map is the identity:

$$\begin{array}{c} G \\ \curvearrowleft \\ H \end{array} = \begin{array}{c} H \\ \downarrow \\ G \end{array}$$

The bimodule map $({}_H G) \longrightarrow ({}_H G {}_H G) \longrightarrow ({}_H G)$ for the left hand side of the first equation in (23) is given by

$$g \longmapsto \sum_{i=1}^m g g_i^{-1} \otimes g_i \longmapsto \sum_{i=1}^m p_H(g g_i^{-1}) g_i.$$

Notice that $p_H(g g_i^{-1}) = 0$ iff $g \notin H g_i$, and $p_H(h g_i g_i^{-1}) = h$. Therefore, $g \longmapsto g$ under the map, and the first equation in (23) holds:

$$\begin{array}{c} H \\ \curvearrowright \\ G \end{array} = \begin{array}{c} H \\ \downarrow \\ G \end{array}$$

The bimodule map $(G_H) \longrightarrow (G_H G_H) \longrightarrow (G_H)$ for the left hand side of the second equation in (23) is given by

$$g \longmapsto \sum_{i=1}^m g_i^{-1} \otimes g_i g \longmapsto \sum_{i=1}^m g_i^{-1} p(g_i g) = g$$

by a similar computation, so that

$$\begin{array}{c} G \\ \curvearrowleft \\ H \end{array} = \begin{array}{c} G \\ \uparrow \\ H \end{array},$$

and the four bimodule maps above determine biadjointness morphisms for induction and restriction functors between H and G .

Consider the \mathbb{k} -algebra

$$\mathbb{k}G^H := \{a \in \mathbb{k}G \mid ha = ah \ \forall h \in H\}$$

of H -invariants in $\mathbb{k}G$ with respect to the conjugation action. This algebra is canonically isomorphic to the endomorphism ring of the bimodule $({}_H G)$, and, therefore, to the endomorphism ring of the functor Res_G^H , via the map that assigns to $a \in \mathbb{k}G^H$ the endomorphism $'a(x) := ax$, where $x \in ({}_H G)$. Likewise, the opposite algebra of $\mathbb{k}G^H$ is canonically isomorphic to the endomorphism ring of the bimodule (G_H) and, therefore, to that of the functor Ind_H^G , via the map that assigns to $a \in \mathbb{k}G^H$ the endomorphism $a'(x) := xa$, where $x \in (G_H)$.

Thus, to $a \in \mathbb{k}G^H$ we assign endomorphisms a' and $'a$ of Ind_H^G and Res_G^H and depict them by

$$\begin{array}{ccc} & \uparrow & \\ G & \bullet & a \quad H \\ & \downarrow & \end{array} \qquad \begin{array}{ccc} & \downarrow & \\ H & a \bullet & G \\ & \uparrow & \end{array}$$

Lemma 1 *For any $a \in \mathbb{k}G^H$ there are equalities of bimodule homomorphisms*

$$\begin{array}{c} G \\ \downarrow \\ a \bullet \quad H \\ \downarrow \\ a \bullet \quad G \\ \downarrow \\ H \end{array} = \begin{array}{c} G \\ \downarrow \\ H \bullet \quad a \\ \downarrow \\ G \bullet \quad a \\ \downarrow \\ H \end{array} \quad \alpha_H^G(a' \circ \text{Id}) = \alpha_H^G(\text{Id} \circ 'a), \quad (25)$$

$$\begin{array}{c} G \\ \downarrow \\ a \bullet \quad H \\ \downarrow \\ a \bullet \quad G \\ \downarrow \\ H \end{array} = \begin{array}{c} G \\ \downarrow \\ H \bullet \quad a \\ \downarrow \\ G \bullet \quad a \\ \downarrow \\ H \end{array} \quad ('a \circ \text{Id})\beta_H^G = (\text{Id} \circ a')\beta_H^G, \quad (26)$$

$$\begin{array}{c} H \\ \downarrow \\ G \bullet \quad a \\ \downarrow \\ a \bullet \quad G \\ \downarrow \\ H \end{array} = \begin{array}{c} H \\ \downarrow \\ a \bullet \quad G \\ \downarrow \\ G \bullet \quad a \\ \downarrow \\ H \end{array} \quad \bar{\alpha}_G^H('a \circ \text{Id}) = \bar{\alpha}_G^H(\text{Id} \circ a'), \quad (27)$$

$$\begin{array}{c} H \\ \downarrow \\ a \bullet \quad G \\ \downarrow \\ a \bullet \quad H \\ \downarrow \\ G \end{array} = \begin{array}{c} H \\ \downarrow \\ G \bullet \quad a \\ \downarrow \\ G \bullet \quad a \\ \downarrow \\ H \end{array} \quad (a' \circ \text{Id})\bar{\beta}_G^H = (\text{Id} \circ 'a)\bar{\beta}_G^H. \quad (28)$$

The left hand side of the first equality is a map $(G_H G) \longrightarrow (G)$ given by $g_1 \otimes g_2 \longmapsto g_1 a \otimes g_2 \longmapsto g_1 a g_2$. The right hand side is $g_1 \otimes g_2 \longmapsto g_1 \otimes a g_2 \longmapsto g_1 a g_2$, and the equality is obvious.

The second equality follows from an equally trivial computation.

The third equality is the equation $p_H(ga) = p_H(ag)$ for $g \in G$ and $a \in \mathbb{k}G^H$. It suffices to check it when $\mathbb{k} = \mathbb{Z}$ and $a = \sum_{h \in H} hkh^{-1}$ for some $k \in G$. The equation becomes

$$\sum_{h \in H} p_H(ghkh^{-1}) = \sum_{h \in H} p_H(hkh^{-1}g) \quad (29)$$

The left hand side equals

$$\sum_{h \in H} p_H(ghk)h^{-1} = \sum_{h^{-1}g^{-1}u=k} p_H(u)p_H(h^{-1}) = \sum_{hg^{-1}u=k} p_H(u)p_H(h)$$

where in the first equality we set $u = ghk$, the sum being over all $u, h \in G$ with $h^{-1}g^{-1}u = k$. For the second equality, we converted h to h^{-1} .

The right hand side of (29) equals

$$\sum_{h \in H} hp_H(kh^{-1}g) = \sum_{ug^{-1}h=k} p_H(h)p_H(u),$$

where we set $u = kh^{-1}g$ and the sum is over all $u, h \in G$. Interchanging h and u , we see that (29) holds.

For the last of the four equations, it suffices to check that the image of $1 \in G$ is the same under these two bimodule homomorphisms. For the one on the left,

$$1 \mapsto \sum_{i=1}^m g_i^{-1} \otimes g_i \mapsto \sum_{i=1}^m g_i^{-1} \otimes ag_i.$$

For the one on the right,

$$1 \mapsto \sum_{i=1}^m g_i^{-1} \otimes g_i \mapsto \sum_{i=1}^m g_i^{-1}a \otimes g_i.$$

Again, we can assume $\mathbb{k} = \mathbb{Z}$ and $a = \sum_{h \in H} hkh^{-1}$ for some $k \in G$. The equation becomes

$$\sum_{i,h} g_i^{-1} \otimes hkh^{-1}g_i = \sum_{i,h} g_i^{-1}hkh^{-1} \otimes g_i, \quad (30)$$

summing over $1 \leq i \leq m$ and $h \in H$. We have

$$\sum_{i,h} g_i^{-1} \otimes hkh^{-1}g_i = \sum_{i,h} g_i^{-1}h \otimes kh^{-1}g_i = \sum_{u \in G} u \otimes ku^{-1},$$

where, in the first equality, h is moved to the left (the tensor product is over $\mathbb{k}H$), and in the second equality $u = g_i^{-1}h$ runs over all elements of G as i changes from 1 to m and h runs over all elements of H . Likewise,

$$\sum_{i,h} g_i^{-1}hkh^{-1} \otimes g_i = \sum_{i,h} g_i^{-1}hk \otimes h^{-1}g_i = \sum_{u \in G} uk \otimes u^{-1}.$$

Equation (30) and Lemma 1 follow.

Since the box labelled by $a \in \mathbb{k}G^H$ can be dragged through any U -turn, we see that the biadjointness maps have the cyclic property – dragging the box labelled a all the way along a circle brings us back to the original diagram. This concludes the proof of Theorem 2. \square

There are obvious simplification relations

$$\begin{array}{c} \text{bubble with } G \text{ inside and } H \text{ outside} \end{array} \stackrel{H}{=} H \qquad G \begin{array}{c} \text{bubble with } H \text{ inside and } G \text{ outside} \end{array} = [G:H] G$$

The first relation says that a counterclockwise bubble with G inside and H outside can be erased. The second relation allows to remove a clockwise bubble at the cost of multiplying the diagram by the index of H in G .

For each inclusion of finite groups $H \subset K \subset G$ there is a canonical isomorphism between induction functors $\text{Ind}_H^G \cong \text{Ind}_K^G \circ \text{Ind}_H^K$ which corresponds to a canonical isomorphism of bimodules $(G)_H \cong (G)_K(K)_H$. Likewise, canonical isomorphism between restrictions $\text{Res}_G^H \cong \text{Res}_K^H \circ \text{Res}_G^K$ is given by a natural isomorphism of bimodules ${}_H(G) \cong {}_H(K)({}_K(G))$. We draw these isomorphisms via trivalent diagrams

$$\begin{array}{c} \text{trivalent vertex with } G \text{ up, } H \text{ right, } K \text{ down} \end{array} \quad \begin{array}{c} \text{trivalent vertex with } K \text{ up, } G \text{ left, } H \text{ right} \end{array} \quad \begin{array}{c} \text{trivalent vertex with } H \text{ up, } G \text{ right, } K \text{ down} \end{array} \quad \begin{array}{c} \text{trivalent vertex with } K \text{ up, } H \text{ left, } G \text{ right} \end{array}$$

Since the isomorphisms are mutually-inverse, we have, for the first two isomorphisms,

$$\begin{array}{c} \text{trivalent vertex with } K \text{ up, } G \text{ left, } H \text{ right} \end{array} = \begin{array}{c} \text{curved lines } G \text{ and } H \text{ meeting at } K \end{array} \qquad \begin{array}{c} \text{trivalent vertex with } G \text{ up, } H \text{ left, } K \text{ right} \end{array} = \begin{array}{c} \text{curved lines } G \text{ and } H \text{ meeting at } K \end{array}$$

Mutual inversion of the other two isomorphisms can be similarly depicted. These definitions are compatible with isotopies-identities

$$\begin{array}{c} \text{trivalent vertex with } G \text{ up, } H \text{ left, } K \text{ right} \end{array} = \begin{array}{c} \text{trivalent vertex with } G \text{ up, } H \text{ left, } K \text{ right} \end{array} = \begin{array}{c} \text{trivalent vertex with } G \text{ up, } H \text{ left, } K \text{ right} \end{array}$$

hold (likewise for the other pair of isomorphisms). These identities imply that various definitions of trivalent vertices in other positions relative to the y -coordinate are all the same. For instance, if we define

$$\begin{array}{c} \text{trivalent vertex with } G \text{ up, } H \text{ left, } K \text{ right} \end{array} := \begin{array}{c} \text{trivalent vertex with } G \text{ up, } H \text{ left, } K \text{ right} \end{array}$$

then

The associativity relation for the inclusions of four groups $L \subset H \subset K \subset G$ has the form

The induction and restriction functors for H and G depend on the inclusion $\iota : H \hookrightarrow G$. Conjugating the inclusion by an element $g \in G$, so that $\iota'(h) = ghg^{-1}$, $\iota' : H \hookrightarrow G$, leads to induction and restriction functors isomorphic to the original ones, via bimodule maps

$$\begin{aligned} (G)_{\iota(H)} &\longrightarrow (G)_{\iota'(H)}, & f &\longmapsto fg^{-1}, & f &\in G, \\ \iota(H)(G) &\longrightarrow \iota'(H)(G), & f &\longmapsto gf, & f &\in G. \end{aligned}$$

We depict these conjugation isomorphisms via a mark on a line with g next to it:

The Mackey induction-restriction theorem says that, given subgroups H, K of a finite group G , there is an isomorphism

$$\text{Res}_G^K \circ \text{Ind}_H^G \cong \bigoplus_{i \in I} \text{Ind}_{K \cap g_i H g_i^{-1}}^K \circ \text{Res}_H^{K \cap g_i H g_i^{-1}}, \quad (31)$$

where the sum is over representatives g_i of (K, H) -cosets of G ,

$$G = \sqcup_{i \in I} K g_i H.$$

Let $L_i = K \cap g_i H g_i^{-1}$. Diagrams

define (K, H) -bimodule maps

$$\alpha_i : (K)_{L_i}'(H) \longrightarrow {}_K(G)_H, \quad \beta_i : {}_K(G)_H \longrightarrow (K)_{L_i}'(H).$$

Here $(K)_{L_i}'(H)$ is $\mathbb{k}[K] \otimes_{\mathbb{k}[L_i]} \mathbb{k}[H]$, with $x \in L_i$ acting on H by right multiplication by $g_i^{-1}xg_i$.

Proposition 6 *The maps $\sum_{i \in I} \alpha_i$ and $\sum_{i \in I} \beta_i$ are mutually-inverse isomorphisms of bimodules $\oplus_{i \in I} (K)_{L_i}'(H)$ and ${}_K(G)_H$.*

This proposition is a pictorial restatement of the Mackey theorem. Proof is left to the reader and amounts to checking the following relations

$$\begin{array}{c}
 \begin{array}{ccc}
 \begin{array}{c} \curvearrowright \\ K \end{array} & G & \begin{array}{c} \curvearrowleft \\ H \end{array} \\
 & & \\
 & & \\
 & & \\
 \end{array} = \sum_{i \in I} \begin{array}{ccc}
 \begin{array}{c} \curvearrowright \\ K \end{array} & \begin{array}{c} G \\ \text{---} \\ L_i \end{array} & \begin{array}{c} \curvearrowleft \\ H \end{array} \\
 & & \\
 & & \\
 & & \\
 \end{array} \\
 \\
 \begin{array}{ccc}
 \begin{array}{c} \curvearrowright \\ K \end{array} & L_i & \begin{array}{c} \curvearrowleft \\ H \end{array} \\
 & & \\
 & & \\
 & & \\
 \end{array} = \begin{array}{ccc}
 \begin{array}{c} \curvearrowright \\ K \end{array} & \begin{array}{c} L_i \\ \text{---} \\ G \end{array} & \begin{array}{c} \curvearrowleft \\ H \end{array} \\
 & & \\
 & & \\
 & & \\
 \end{array} \\
 \\
 \begin{array}{ccc}
 \begin{array}{c} \curvearrowright \\ K \end{array} & \begin{array}{c} L_j \\ \text{---} \\ G \end{array} & \begin{array}{c} \curvearrowleft \\ H \end{array} \\
 & & \\
 & & \\
 & & \\
 \end{array} = 0 \quad \text{if } i \neq j
 \end{array}$$

3.3 Induction and restriction between symmetric groups

We now specialize the earlier construction to the case of the symmetric group S_n , viewed as the permutation group of $\{1, 2, \dots, n\}$, and induction/restriction functors for inclusions $S_n \subset S_{n+1}$, where S_n is identified with the stabilizer of $n+1$ in S_{n+1} . Notations for bimodules will be further simplified, so that, for instance, ${}_n(n+1)_{n-1}$ stands for $\mathbb{k}[S_{n+1}]$, viewed as a $(\mathbb{k}[S_n], \mathbb{k}[S_{n-1}])$ -bimodule for the standard inclusions $S_n \subset S_{n+1} \supset S_{n-1}$, and ${}_n(n+1)_n(n+2)$ stands for $\mathbb{k}[S_{n+1}] \otimes_{\mathbb{k}[S_n]} \mathbb{k}[S_{n+2}]$, viewed as a $(\mathbb{k}[S_n], \mathbb{k}[S_{n+2}])$ -bimodule. The regions of the strip $\mathbb{R} \times [0, 1]$ are now labelled by nonnegative integers n . An upward oriented line separating regions labelled n and $n+1$

$$\begin{array}{ccc} & \uparrow & \\ \mathbf{n+1} & & \mathbf{n} \end{array}$$

denotes the identity endomorphism of the induction functor

$$\mathrm{Ind}_n^{n+1} : \mathbb{k}[S_n]\text{-mod} \longrightarrow \mathbb{k}[S_{n+1}]\text{-mod}.$$

This is the functor of tensoring with the bimodule $(n+1)_n$.

A downward oriented line separating regions $n+1$ and n

$$\begin{array}{ccc} & \downarrow & \\ \mathbf{n} & & \mathbf{n+1} \end{array}$$

denotes the identity endomorphism of the restriction functor

$$\mathrm{Res}_{n+1}^n : \mathbb{k}[S_{n+1}]\text{-mod} \longrightarrow \mathbb{k}[S_n]\text{-mod}.$$

The bimodule for this functor is ${}_n(n+1)$.

The four U -turns are given by the following bimodule maps:

$$\begin{array}{ccc} \begin{array}{c} \textcircled{\mathbf{n}}^{\mathbf{n+1}} \\ \searrow \end{array} & (n+1)_n(n+1) \longrightarrow (n+1), \\ & g \otimes h \longmapsto gh, \quad g, h \in S_{n+1}, \end{array} \quad (32)$$

$$\begin{array}{ccc} \begin{array}{c} \textcircled{\mathbf{n+1}}^{\mathbf{n}} \\ \swarrow \end{array} & (n) \longrightarrow {}_n(n+1)_n, \quad g \longmapsto g, \quad g \in S_n \end{array} \quad (33)$$

$$\begin{array}{ccc} \begin{array}{c} \textcircled{\mathbf{n+1}}^{\mathbf{n}} \\ \swarrow \end{array} & p_n : {}_n(n+1)_n \longrightarrow (n), \quad p_n(g) = \begin{cases} g & \text{if } g \in S_n, \\ 0 & \text{otherwise} \end{cases} \end{array} \quad (34)$$

$$\begin{array}{ccc} \begin{array}{c} \textcircled{\mathbf{n}}^{\mathbf{n+1}} \\ \swarrow \end{array} & q_n : (n+1) \longrightarrow (n+1)_n(n+1), \end{array} \quad (35)$$


where the bimodule map q_n is determined by the condition


$$q_n(1) = \sum_{i=1}^{n+1} s_i s_{i+1} \dots s_n \otimes s_n \dots s_{i+1} s_i, \quad s_i = (i, i+1).$$


Notice that $\{s_n \dots s_2 s_1, s_n \dots s_3 s_2, \dots, s_n s_{n-1}, s_n, 1\}$ are $n+1$ coset representatives of $S_n \subset S_{n+1}$, and


$$\begin{aligned} q_n(g) &= \sum_{i=1}^{n+1} g s_i s_{i+1} \dots s_n \otimes s_n \dots s_{i+1} s_i \\ &= \sum_{i=1}^{n+1} s_i s_{i+1} \dots s_n \otimes s_n \dots s_{i+1} s_i g, \quad g \in S_{n+1}. \end{aligned}$$

Bimodule maps p_n and q_n are the second adjointness maps for the group inclusion $S_n \subset S_{n+1}$. For an arbitrary inclusion of finite groups $H \subset G$ these maps were described in the previous subsection.

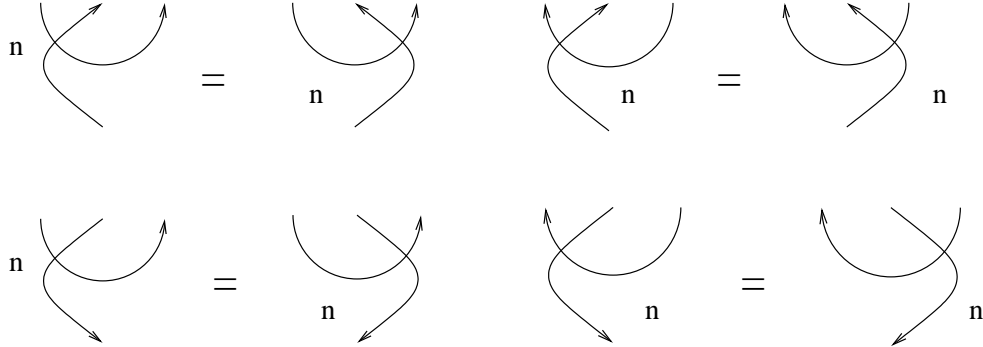
Denote by an upward-pointing crossing  \mathbf{n} the endomorphism of $(n+2)_n$ given by right multiplication by s_{n+1} , so that $g \mapsto gs_{n+1}, g \in S_{n+2}$.

Denote by a downward-pointing crossing  \mathbf{n} the endomorphism of ${}_n(n+2)$ given by left multiplication by s_{n+1} , so that $g \mapsto s_{n+1}g, g \in S_{n+2}$.

Denote by a right-pointing crossing  \mathbf{n} the bimodule endomorphism $(n)_{n-1}(n) \rightarrow {}_n(n+1)_n$ that takes $g \otimes h$ for $g, h \in S_n$ to $gs_nh \in S_{n+1}$.

Denote by a left-pointing crossing  \mathbf{n} the bimodule endomorphism ${}_n(n+1)_n \rightarrow (n)_{n-1}(n)$ that takes $g \in S_n \subset S_{n+1}$ to 0 and gs_nh for $g, h \in S_n$ to $g \otimes h \in (n)_{n-1}(n)$.

These four definitions-notations are compatible with the isotopies of diagrams in the plane strip—there are equalities of bimodule endomorphisms



They can be checked by direct computations.

Proposition 7 *The following relations hold for any $n \in \mathbb{Z}$.*

$$\begin{array}{lcl}
\begin{array}{c} \nearrow \\ \searrow \\ \nearrow \end{array} n = \begin{array}{c} \uparrow \\ \uparrow \end{array} n & \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \end{array} n = \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \end{array} n \\
\begin{array}{c} \nwarrow \\ \swarrow \\ \nwarrow \end{array} n = \begin{array}{c} \uparrow \\ \downarrow \end{array} n & \begin{array}{c} \circlearrowleft \end{array} n = 1 \\
\begin{array}{c} \nearrow \\ \searrow \\ \nwarrow \end{array} n = \begin{array}{c} \downarrow \\ \uparrow \end{array} n - \begin{array}{c} \smile \\ \frown \end{array} n & \begin{array}{c} \circlearrowright \end{array} n = 0
\end{array} \tag{36}$$

Here and further we follow the convention that a diagram equal 0 if it has a region labelled by a negative number. Notice that the top two relations come from the relations in the symmetric groups: $s_{n+1}^2 = 1$ and $s_{n+1}s_{n+2}s_{n+1} = s_{n+2}s_{n+1}s_{n+2}$ and follow at once from the definition of the bimodule homomorphism associated to an upward-pointing crossing. The four remaining relations encode the bimodule decomposition

$${}_n(n+1)_n \cong (n)_{n-1}(n) \oplus (n)$$

giving an isomorphism

$$\text{Res}_{n+1}^n \circ \text{Ind}_n^{n+1} \cong \text{Ind}_{n-1}^n \circ \text{Res}_n^{n-1} \oplus \text{Id}$$

of endofunctors in the category $\mathbb{k}[S_n]\text{-mod}$. This is a special case of the Mackey decomposition theorem which was given a diagrammatic interpretation in an earlier section. Proposition 7 relations are identical to the ones in the definition of the category \mathcal{H}' , see Section 2.1.

Let \mathcal{S}' be the category whose objects are compositions of induction and restriction functors between symmetric groups for standard embeddings $S_k \subset S_{k+1}$. The morphisms are natural transformations of functors (again, we work over a field \mathbb{k} of characteristic 0). The category \mathcal{S}' is the sum of categories \mathcal{S}'_k over $k \geq 0$; in the latter the first induction or restriction starts from S_k . For instance, $\text{Ind}_{k+1}^{k+2} \circ \text{Ind}_k^{k+1} \circ \text{Res}_{k+1}^k \circ \text{Ind}_k^{k+1}$ is an object of \mathcal{S}'_k . Morphisms in \mathcal{S}'_k are natural transformation of functors and can be identified with homomorphisms of associated bimodules.

Thus, for each $k \geq 0$, there is a functor $\mathcal{F}'_k : \mathcal{H}' \rightarrow \mathcal{S}'_k$ that takes Q_ϵ to the corresponding composition of induction and restriction functors. For instance,

$$\mathcal{F}'_k(Q_{++-+}) = \text{Ind}_{k+1}^{k+2} \circ \text{Ind}_k^{k+1} \circ \text{Res}_{k+1}^k \circ \text{Ind}_k^{k+1}.$$

If, for some m , the last m terms of ϵ contain at least $k+1$ more minuses than pluses, then $\mathcal{F}'_k(Q_\epsilon) = 0$. On morphisms \mathcal{F}'_k is defined as follows. It takes a diagram representing a

morphism in \mathcal{H}' , labels the rightmost region of the diagram by k , and views the diagram as a natural transformation between compositions of induction and restriction functors. The functor \mathcal{F}'_k is not monoidal, since \mathcal{S}'_k does not have a monoidal structure matching that of \mathcal{H}' .

Let \mathcal{S} , respectively \mathcal{S}_k , be the Karoubi envelope of \mathcal{S}' , respectively \mathcal{S}'_k . Functor \mathcal{F}'_k induces a functor on Karoubi envelopes $\mathcal{F}_k : \mathcal{H} \rightarrow \mathcal{S}_k$. We summarize relevant categories and functors below.

$$\mathcal{S} = \text{Kar}(\mathcal{S}'), \quad \mathcal{S}_k = \text{Kar}(\mathcal{S}'_k), \quad \mathcal{F}'_k : \mathcal{H}' \rightarrow \mathcal{S}'_k, \quad \mathcal{F}_k : \mathcal{H} \rightarrow \mathcal{S}_k.$$

$$\begin{array}{ccc} \mathcal{S}' & \xlongequal{\quad} & \bigoplus_{k \geq 0} \mathcal{S}'_k \\ \text{Kar} \downarrow & & \downarrow \text{Kar} \\ \mathcal{S} & \xlongequal{\quad} & \bigoplus_{k \geq 0} \mathcal{S}_k \end{array} \quad \begin{array}{ccc} \mathcal{H}' & \xrightarrow{\mathcal{F}'_k} & \mathcal{S}'_k \\ \text{Kar} \downarrow & & \downarrow \text{Kar} \\ \mathcal{H} & \xrightarrow{\mathcal{F}_k} & \mathcal{S}_k \end{array}$$

Functor \mathcal{F}_k induces a homomorphism of abelian groups

$$[\mathcal{F}_k] : K_0(\mathcal{H}) \rightarrow K_0(\mathcal{S}_k). \quad (37)$$

Notice that $K_0(\mathcal{H})$ is a ring, while $K_0(\mathcal{S}_k)$ is only an abelian group.

An object of \mathcal{S}_k is a direct summand of a finite sum of composition of induction and restriction functors that start with the category of $\mathbb{k}[S_k]$ -modules, thus it takes any finite-dimensional $\mathbb{k}[S_k]$ -module to a module over $\bigoplus_{m \geq 0} \mathbb{k}[S_m]$. Descending to Grothendieck groups, we obtain a homomorphism

$$\theta_k : K_0(\mathcal{S}_k) \rightarrow \text{Hom}_{\mathbb{Z}}(K_0(\mathbb{k}[S_k]), \bigoplus_{m \geq 0} K_0(\mathbb{k}[S_m])).$$

From here until the end of this paper we assume that $\text{char}(\mathbb{k}) = 0$. Consider the composite homomorphism

$$\theta_k[\mathcal{F}_k] : K_0(\mathcal{H}) \rightarrow \text{Hom}_{\mathbb{Z}}(K_0(\mathbb{k}[S_k]), \bigoplus_{m \geq 0} K_0(\mathbb{k}[S_m])).$$

This homomorphism takes $[Q_{+, \mu}]$ to a map that assigns to $[M] \in K_0(\mathbb{k}[S_k])$, for a $\mathbb{k}[S_k]$ -module M , the symbol $[\text{Ind}_{S_{|\mu|} \times S_k}^{S_{|\mu|+k}}(L_{\mu} \otimes M)]$ of induced module over $\mathbb{k}[S_{|\mu|+k}]$. In other words, tensor M with L_{μ} , producing a module over $\mathbb{k}[S_{|\mu|}] \times S_k$, induce to $\mathbb{k}[S_{|\mu|+k}]$, then pass to the Grothendieck group.

Likewise, $\theta_k[\mathcal{F}_k]$ takes $[Q_{-, \lambda}]$ to the zero map if $|\lambda| > k$ and, if $k \geq |\lambda|$, to the map that assigns to $[M]$ as above the symbol of the module

$$\text{Hom}_{\mathbb{k}[S_{|\lambda|}]}(L_{\lambda}, M) \in \mathbb{k}[S_{k-|\lambda|}]\text{-mod}$$

In other words, restrict M to being a module over the group algebra of $S_{|\lambda|} \times S_{k-|\lambda|} \subset S_k$, and form homs from the simple module L_λ over $S_{|\lambda|}$. The result is a representation of the symmetric group $S_{k-|\lambda|}$.

Now consider composition

$$\theta_k[\mathcal{F}_k]\gamma : H_{\mathbb{Z}} \longrightarrow \text{Hom}_{\mathbb{Z}}(K_0(\mathbb{k}[S_k]), \bigoplus_{m \geq 0} K_0(\mathbb{k}[S_m])).$$

We claim that the sum of these maps, over all $k \geq 0$, is injective. Let

$$y = \sum_{\lambda, \mu} y_{\lambda, \mu} b_\mu a_\lambda, \quad y_{\lambda, \mu} \in \mathbb{Z}$$

be an arbitrary nonzero element of $H_{\mathbb{Z}}$. We have

$$\gamma(y) = \sum_{\lambda, \mu} y_{\lambda, \mu} [Q_{+, \mu^*}] [Q_{-, \lambda}].$$

When $|\lambda| = k$, the map

$$\theta_k[\mathcal{F}_k]\gamma(a_\lambda) = \theta_k[\mathcal{F}_k]([Q_{-, \lambda}])$$

takes $[L_\nu]$ to 0 if $|\nu| = k$ and $\nu \neq \lambda$. The same map takes $[L_\lambda]$ to $[L_\emptyset]$, the symbol of the simple module over $\mathbb{k}[S_0] = \mathbb{k}$.

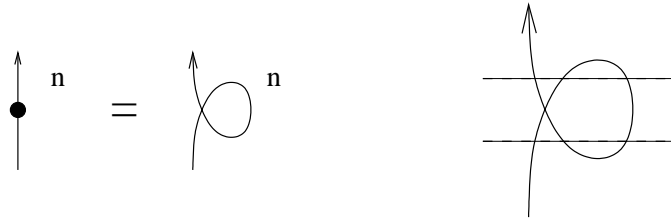
Choose k such that $y_{\lambda, \mu} \neq 0$ for some λ with $|\lambda| = k$ and some μ , while $y_{\lambda, \mu} = 0$ for all μ whenever $|\lambda| < k$. Also choose ν with $|\nu| = k$ and $y_{\nu, \mu} \neq 0$ for some μ . The map $\theta_k[\mathcal{F}_k]\gamma(y)$ takes $[L_\nu]$ to

$$\sum_{\mu} y_{\nu, \mu} [L_{\mu^*}] \neq 0.$$

Therefore, $\theta_k[\mathcal{F}_k]\gamma(y)$ is a nonzero map, and $\gamma(y) \neq 0$. This concludes the proof that γ is injective (Theorem 1).

4 The size of morphism spaces in \mathcal{H}'

In this section we will prove Propositions 3 and 4. Consider the right curl with the rightmost region labelled n (also recall the shorthand of denoting this curl by a dot). This curl can be realized as the composition of a cup with a crossing with a cap.



The corresponding endomorphism of the bimodule $(n+1)_n$ takes 1 to

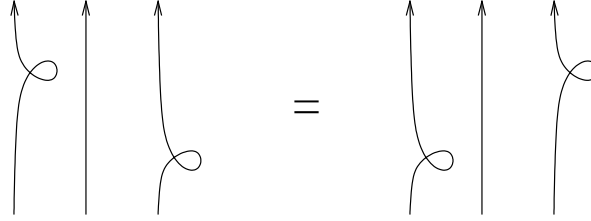
$$J_n := \sum_{i=1}^n s_i \dots s_{n-1} s_n s_{n-1} \dots s_i = (1, n+1) + (2, n+1) + \dots + (n, n+1).$$

This endomorphism of $(n+1)_n$ is the right multiplication by J_n :

$$g \longmapsto gJ_n, \quad g \in S_{n+1}.$$

Notice that J_n is the Jucys-Murphy element, ubiquitous in the representation theory of the symmetric group. Our diagrammatics realizes it via the right curl and interprets the endomorphism of multiplication by J_n as the composition of three natural transformations, two of which (cup and cap) come from biadjointness of the induction and restriction functors.

Commutativity of Jucys-Murphy elements now acquires a graphical interpretation as isotopies of curls (or dots) on upward-oriented disjoint strands past each other.



For each $n \geq 0$ the functor \mathcal{F}'_n , applied to $\mathbf{1}$ and its endomorphisms, produces a homomorphism from $\text{End}_{\mathcal{H}'}(\mathbf{1})$ to $Z(\mathbb{k}[S_n])$, the center of the group ring of the symmetric group. Composing with the homomorphism ψ_0 , we obtain the homomorphism

$$\psi_{0,n} : \Pi \cong \mathbb{k}[c_0, c_1, \dots] \longrightarrow Z(\mathbb{k}[S_n])$$

that takes c_k to

$$c_{k,n} = \sum_{i=1}^n s_i \dots s_{n-1} J_{n-1}^k s_{n-1} \dots s_i \tag{38}$$

$$= \sum_{i=1}^n (i, i+1, \dots, n) ((1, n) + \dots + (n-1, n))^k (n, n-1, \dots, i). \tag{39}$$

We'd like to show that the union of $\psi_{0,n}$ over all n ,

$$\Pi \longrightarrow \bigoplus_{n \geq 0} Z(\mathbb{k}[S_n]),$$

is injective. For the first few values of k we have

$$\begin{aligned} c_{0,n} &= n, \\ c_{1,n} &= 2 \sum_{1 \leq i < j \leq n} (i, j), \\ c_{2,n} &= 3 \sum (i_1, i_2, i_3) + n(n-1), \end{aligned}$$

where the sum in the bottom line is over all 3-cycles. In general,

$$c_{k,n} = (k+1) \sum (i_1, \dots, i_{k+1}) + \text{l.o.t.}, \quad (40)$$

where the sum is over all $k+1$ -cycles. For $k > 0$, the lower order terms is a sum over permutations of disturbance less than $k+1$, where we define the disturbance of a permutation $\sigma \in S_n$ as the number of elements moved by σ ,

$$\text{dist}(\sigma) = |\{i | 1 \leq i \leq n, \sigma(i) \neq i\}|.$$

Notice that $c_{k,n}$ is the sum of conjugates of J_{n-1}^k and contains terms of disturbance at most $k+1$, for $k > 0$.

Since $\text{char}(\mathbb{k}) = 0$, the coefficient of the sum of $(k+1)$ -cycles in (40) is nonzero. Consider an increasing filtration

$$Z_0 = \mathbb{k} \subset Z_2 \subset Z_3 \subset \dots \subset Z_n = \mathbb{Z}(\mathbb{k}[S_n])$$

by having Z_i be the span over conjugacy classes that consist of permutations of disturbance at most i . We make $\mathbb{k}[c_0, c_1, \dots]$ graded by $\deg(c_0) = 0$ and $\deg(c_k) = k+1$ for $k > 0$, and then consider associated increasing filtration on $\mathbb{k}[c_0, c_1, \dots]$, where the i -th filtered terms is the sum of graded terms of degree at most i . The homomorphism $\psi_{0,n}$ respects the two filtrations.

To show asymptotic faithfulness of $\psi_{0,n}$ as $n \rightarrow \infty$ we examine induced homomorphism of adjoint graded rings. Assume that there is a universal relation in $\text{End}_{\mathcal{H}}(\mathbf{1})$

$$\sum_I a_I c_I = 0$$

which holds for all sufficiently large n . Here $I = (a_0, a_1, \dots, a_i)$ is a finite sequence of non-negative integers, $a_I \in \mathbb{k}$, and $c_I = c_0^{a_0} c_1^{a_1} \dots c_i^{a_i}$ is the corresponding monomial in $\mathbb{k}[c_0, c_1, \dots]$. The sum is over finitely many sequences I . This implies relations

$$\sum_I a_I c_{I,n} = 0$$

for all n , where $c_{I,n} = c_{0,n}^{a_0} c_{1,n}^{a_1} \dots c_{i,n}^{a_i}$. Choose any term from the sum corresponding to $I = (a_0, a_1, \dots, a_i)$ with the highest possible degree monomial c_I . The only terms in

the sum $\sum_I a_I c_{I,n}$ that contribute to the conjugacy class of type $(a_1 + 1, \dots, a_i + 1)$ can come from sequences $I' = (a, a_1, \dots, a_i)$ that differ from I only in the first term. Since, for such I' ,

$$c_{I',n} = n^a c_{1,n}^{a_1} \dots c_{i,n}^{a_i}$$

the relation $\sum_I a_I c_{I,n} = 0$ implies $\sum_a a_{I'} n^a = 0$, which, in turn, leads to $a_{I'} = 0$ for all I' as above, since the only polynomial with infinitely many positive integers n as roots is the zero polynomial. This contradiction implies asymptotic faithfulness of $\psi_{0,n}$. Therefore, ψ_0 is injective, which concludes the proof of Proposition 3 when $\text{char } \mathbb{k} = 0$. Take $\mathbb{k} = \mathbb{Q}$, then pass to the subring \mathbb{Z} of \mathbb{Q} . Our arguments imply that ψ_0 is an isomorphism over \mathbb{Z} , and, hence, over any commutative ring \mathbb{k} , including any field.

We next prove injectivity of ψ_m (Proposition 4), see equation (18), again first in the characteristic zero case. The algebra $DH_m \otimes \Pi$ has a basis of elements $x_1^{a_1} \dots x_m^{a_m} \cdot \sigma \cdot c_0^{b_0} c_1^{b_1} \dots c_k^{b_k}$ over permutations $\sigma \in S_m$ and $a_i, b_j \in \mathbb{Z}_+$, see discussion before Proposition 4.

Applying functor \mathcal{F}'_n to Q_{+m} and its endomorphism ring gives us a homomorphism

$$\text{End}_{\mathcal{H}'}(Q_{+m}) \longrightarrow \text{End}((n+m)_n)$$

to the endomorphism ring of the $(\mathbb{k}[S_{n+m}], \mathbb{k}[S_n])$ -bimodule $\mathbb{k}[S_{n+m}]$, which we also denote $(n+m)_n$. The composite homomorphism

$$\psi_{m,n} : DH_m \otimes \Pi \longrightarrow \text{End}((n+m)_n)$$

takes elements of $DH_m \otimes \Pi$ to endomorphisms given by right multiplication by suitable elements of $\mathbb{k}[S_{n+m}]$. Namely, $\psi_{m,n}(\sigma)$, for a permutation $\sigma \in S_m \subset DH_m$, is the right multiplication by $\underline{\sigma}$, where we define the latter as the translate of σ by n :

$$\underline{\sigma}(i+n) = \sigma(i) + n, \quad 1 \leq i \leq m, \quad \underline{\sigma}(i) = i, \quad 1 \leq i \leq n.$$

$\psi_{m,n}(x_i)$, where x_i the the diagram of m vertical lines with the dot (right curl) on the i -th strand from the left, is the endomorphism of right multiplication by J_{n+m-i} , and $\psi_{m,n}(c_k)$ is the right multiplication by $c_{k,n}$. The map $\psi_{m,n}$ is described by the corresponding homomorphism $\psi'_{m,n}$ from $DH_m \otimes \Pi$ to the opposite of the group algebra $\mathbb{k}[S_{n+m}]^{op} \supset \text{End}((n+m)_n)$, with $\psi'_{m,n}(x_i) = J_{n+m-i}$, $\psi'_{m,n}(c_k) = c_{k,n}$, etc. We need to take the opposite algebra since the ring of endomorphism of a ring A viewed as a left A -module is naturally isomorphic to the opposite of A : $\text{End}_A({}_A A, {}_A A) \cong A^{op}$.

Define m -disturbance of a permutation $\sigma \in S_{n+m}$ as the number of integers between 1 and n that are moved by σ :

$$\text{dist}_m(\sigma) = |\{i | 1 \leq i \leq n, \sigma(i) \neq i\}|.$$

Notice that $\text{dist}_m(\sigma\tau) \leq \text{dist}_m(\sigma) + \text{dist}_m(\tau)$. On the group algebra $\mathbb{k}[S_{n+m}]$ we can introduce an increasing filtration

$$\mathbb{k}[S_m] = Z_0^m \subset Z_1^m \subset \dots \subset Z_n^m = \mathbb{k}[S_{n+m}]$$

where Z_k^m is spanned by permutations of disturbance at most k .

We turn $DH_m \otimes \Pi$ into a filtered algebra by setting $\deg(c_0) = 0$, $\deg(c_k) = k + 1$ if $k > 0$, $\deg(x_i) = 1$ and $\deg(\sigma) = 0$ and then making the k -th term in the increasing filtration be spanned by the basis elements of total degree at most k .

Homomorphism $\psi'_{m,n}$ is that of filtered algebras, and we can pass to the homomorphism of associated graded algebras. To show asymptotic faithfulness of $\psi'_{m,n}$ we fix m and will be taking n large compared to m . Assume that there exists a relation

$$\sum d_{\sigma, \mathbf{a}, \mathbf{b}} x_1^{a_1} \dots x_m^{a_m} \cdot \sigma \cdot c_0^{b_0} c_1^{b_1} \dots c_r^{b_r} = 0 \quad (41)$$

in $\text{End}_{\mathcal{H}'}(Q_{+^m})$ for some $d_{\sigma, \mathbf{a}, \mathbf{b}} \in \mathbb{k} \setminus \{0\}$, with $\mathbf{a} = (a_1, \dots, a_m)$, $\mathbf{b} = (b_1, \dots, b_r)$, the sum over finitely many triples $(\sigma, \mathbf{a}, \mathbf{b})$. Let

$$x(\sigma, \mathbf{a}, \mathbf{b}) = x_1^{a_1} \dots x_m^{a_m} \cdot \sigma \cdot c_0^{b_0} c_1^{b_1} \dots c_r^{b_r}$$

denote the elements of our basis of $DH_m \otimes \Pi$. The element $\psi'_{m,n}(x(\sigma, \mathbf{a}, \mathbf{b})) \in \mathbb{k}[S_{n+m}]$ belongs to the k -th term of the filtration of the latter, where

$$k = a_1 + \dots + a_m + 2b_1 + 3b_2 + \dots + (r+1)b_r,$$

but not to the $(k-1)$ -st term. Among terms $x(\sigma, \mathbf{a}, \mathbf{b})$ in the sum select only those with the maximal possible k (denote such k by k_0). It's enough to show that, as we sum over only these terms, the image of $\sum \psi'_{m,n}(x(\sigma, \mathbf{a}, \mathbf{b}))$ in the associated graded ring of $\mathbb{k}[S_{n+m}]$ relative to the above filtration is nonzero. In other words, we need to show that coefficients of permutations of disturbance k_0 are not all zero for some sufficiently large n in the expansion of $\psi_{m,n}$ applied to the LHS of (41). This is obtained by looking at the structure of these permutations. They are disjoint unions of cycles, with some of the cycles containing one or more elements of the set $P = \{n+1, \dots, n+m\}$. The relative positions of elements of P in the cycles, lengths of portions of the cycles between elements of P , together with the number of cycles of each length without elements of P uniquely determine σ , a_1, \dots, a_m and c_1, \dots, c_r that can contribute to the coefficient of such permutation. Coefficients at different powers of c_0 are taken care in the same way as in the $m=0$ case. Linear independence of our spanning set of $\text{End}_{\mathcal{H}'}(Q_{+^m})$ and Proposition 4 follow when $\text{char } \mathbb{k} = 0$. Same argument as in the $m=0$ case then implies that ψ_m is an isomorphism over any commutative ring \mathbb{k} .

The formula (40) implies that the natural homomorphism $\text{End}_{\mathcal{H}'}(\mathbf{1}) \longrightarrow Z(\mathbb{k}[S_n])$ from the endomorphisms of the identity object of \mathcal{H}' to the center of the group algebra is surjective when the field \mathbb{k} has characteristic 0. Combining with the result of Cherednik [10] and Olshanskii [37], [8, Theorem 3.2.6] that the centralizer algebra of $\mathbb{k}[S_n]$ in $\mathbb{k}[S_{n+m}]$ is generated by DH_m and the center of $\mathbb{k}[S_n]$, we obtain that the homomorphism $DH_m \otimes \Pi \longrightarrow \text{End}((n+m)_n)$ introduced above is surjective when $\text{char } \mathbb{k} = 0$.

5 Remarks on the Grothendieck ring of \mathcal{H}

5.1 Idempotent rings from \mathcal{H}'

For a sequence ϵ of pluses and minuses denote by $\langle \epsilon \rangle$ the difference between the number of pluses and minuses in ϵ (the weight of ϵ). Then $\text{Hom}_{\mathcal{H}'}(Q_\epsilon, Q_{\epsilon'}) = 0$ if and only if $\langle \epsilon \rangle \neq \langle \epsilon' \rangle$. The "if" part of this observation implies that categories \mathcal{H}' and \mathcal{H} , viewed as additive categories, decompose into the direct sum of subcategories

$$\mathcal{H}' = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{H}'_\ell, \quad \mathcal{H} = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{H}_\ell,$$

where \mathcal{H}'_ℓ is a full subcategory of \mathcal{H}' which contains objects Q_ϵ over all sequences of weight ℓ , and \mathcal{H}_ℓ is the Karoubi envelope of \mathcal{H}'_ℓ .

This direct sum decomposition induces a grading on the Grothendieck ring

$$K_0(\mathcal{H}) = \bigoplus_{\ell \in \mathbb{Z}} K_0(\mathcal{H}_\ell)$$

The Heisenberg algebra H and its integral form $H_{\mathbb{Z}}$ are graded by $\deg(a_n) = n = -\deg(b_n)$, and the homomorphism $\gamma : H_{\mathbb{Z}} \rightarrow K_0(\mathcal{H})$ is that of graded rings.

We can redefine the Grothendieck groups $K_0(\mathcal{H})$ and $K_0(\mathcal{H}_\ell)$ via idempotent rings. For a sequence ϵ let

$$\text{End}(\epsilon) := \text{End}_{\mathcal{H}}(Q_\epsilon)$$

denote the endomorphism algebra of Q_ϵ . Likewise, denote

$$\text{Hom}(\epsilon, \epsilon') := \text{Hom}_{\mathcal{H}}(Q_\epsilon, Q_{\epsilon'}).$$

To the category \mathcal{H} we can assign the idempotent ring of all homomorphisms between various tensor products of generating objects Q_+ and Q_- :

$$\underline{R} := \bigoplus_{\epsilon, \epsilon'} \text{Hom}(\epsilon, \epsilon'),$$

the sum over all sequences ϵ, ϵ' . Ring \underline{R} is nonunital, but has a family of idempotents $1_\epsilon = 1 \in \text{End}(\epsilon)$.

Right projective \underline{R} -modules $1_\epsilon \underline{R}$ correspond to objects Q_ϵ , in the sense that

$$\text{Hom}_{\underline{R}}(1_\epsilon \underline{R}, 1_{\epsilon'} \underline{R}) = \text{Hom}(\epsilon, \epsilon') = \text{Hom}_{\mathcal{H}'}(Q_\epsilon, Q_{\epsilon'}), \quad (42)$$

and the Grothendieck group of finitely-generated projective right \underline{R} -modules is canonically isomorphic to the Grothendieck group of \mathcal{H} :

$$K_0(\underline{R}) \cong K_0(\mathcal{H}).$$

This isomorphism takes $[1_\epsilon \underline{R}]$ to $[Q_\epsilon]$. Usually we use $K_0(A)$ to denote the Grothendieck group of finitely-generated projective left, not right, A -modules. Here, because of (42),

we use right \underline{R} -modules in the definition of $K_0(\underline{R})$. Alternatively, we could use left \underline{R}^{op} -modules, or even left \underline{R} -modules after fixing an isomorphism $\underline{R} \cong \underline{R}^{op}$ (involution ξ_2 induces one such isomorphism). We have

$$\underline{R} = \bigoplus_{\ell \in \mathbb{Z}} \underline{R}_\ell, \quad \underline{R}_\ell := \bigoplus_{\langle \epsilon \rangle = \langle \epsilon' \rangle = \ell} \text{Hom}(\epsilon, \epsilon').$$

Assume from now on that $\ell \geq 0$ (the other case can be treated similarly, or by applying symmetry ξ_3 to reverse ℓ). Given a sequence ϵ with $n + \ell$ pluses and n minuses, the object Q_ϵ of \mathcal{H}_ℓ decomposes into the direct sum of objects $Q_{+^{k+\ell}-k}$ with $0 \leq k \leq n$ with some multiplicities. Hence, \underline{R}_ℓ is Morita equivalent to the idempotented ring

$$R_\ell = \bigoplus_{k, k'=0}^{\infty} \text{Hom}(+^{k+\ell}-k, +^{k'+\ell}-k'),$$

and the inclusion $R_\ell \subset \underline{R}_\ell$ induces an isomorphism of Grothendieck groups $K_0(R_\ell) \cong K_0(\underline{R}_\ell)$. Let

$$R_{\ell, m} = \bigoplus_{k, k'=0}^m \text{Hom}(+^{k+\ell}-k, +^{k'+\ell}-k').$$

The ring R_ℓ is the union of rings in the increasing chain $R_{\ell, 0} \subset R_{\ell, 1} \subset \dots$. Formation of Grothendieck group commutes with direct limits, implying that $K_0(R_\ell)$ is the direct limit of $K_0(R_{\ell, m})$ as m goes to infinity. Thus, there is an isomorphism

$$K_0(\mathcal{H}_\ell) \cong \lim_{m \rightarrow \infty} K_0(R_{\ell, m}).$$

For each k between 0 and m natural inclusion of rings $\text{End}(+^{k+\ell}-k) \subset R_{\ell, m}$ induces a homomorphism of groups $K_0(\text{End}(+^{k+\ell}-k)) \longrightarrow K_0(R_{\ell, m})$. Conjecture 1 would follow from the following two conjectures:

Conjecture 1.1. The standard inclusion $\mathbb{k}[S_n \times S_m] \subset \text{End}(+^n -^m)$ induces an isomorphism of Grothendieck groups of these two rings.

Conjecture 1.2. Ring inclusion

$$\bigoplus_{k=0}^m \text{End}(+^{k+\ell}-k) \subset R_{\ell, m}$$

induces an isomorphism on Grothendieck groups.

We don't know how to prove either statement, but will present now some weak evidence in favor of Conjecture 1.1.

5.2 Grothendieck group of degenerate affine Hecke algebra

Here we prove Conjecture 1.1 in the case $m = 0$ ($n = 0$ case follows by symmetry). By Proposition 4, the endomorphism ring of the object Q_{+^n} of \mathcal{H} is isomorphic to the

tensor product of the degenerate affine Hecke algebra DH_n and the polynomial algebra Π :

$$\text{End}(+^n) \cong DH_n \otimes \Pi.$$

The inclusion $\mathbb{k}[S_n] \hookrightarrow DH_n$ is split, via the homomorphism $\tau_n : DH_n \rightarrow \mathbb{k}[S_n]$ which takes generators s_i of DH_i to transpositions $(i, i+1)$ and generators x_i to Jucys-Murphy elements. The split inclusion induces a split short exact sequence of two rings and an ideal

$$0 \longrightarrow \ker(\tau_n) \longrightarrow DH_n \longrightarrow \mathbb{k}[S_n] \longrightarrow 0,$$

which, in turn, induces a split short exact sequence of K_0 -groups

$$0 \longrightarrow K_0(\ker(\tau_n)) \longrightarrow K_0(DH_n) \longrightarrow K_0(\mathbb{k}[S_n]) \longrightarrow 0,$$

see [40, 42]. Introduce an increasing filtration

$$0 = Z_{-1}DH_n \subset Z_0DH_n \subset Z_1DH_n \subset \dots$$

on DH_n , where Z_kDH_n is spanned by elements of the form $x_1^{a_1} \dots x_n^{a_n} \sigma$ over all $\sigma \in S_n$ and $a_1 + \dots + a_n \leq k$. Then $Z_kDH_n \times Z_mDH_n \subset Z_{k+m}DH_n$ and $Z_0DH_n = \mathbb{k}[S_n]$. Let $B = \text{gr}DH_n$ with respect to this filtration. B is a graded algebra isomorphic to the cross-product of the polynomial algebra in n generators with the group algebra of the symmetric group, $B \cong \mathbb{k}[x_1, \dots, x_n] * \mathbb{k}[S_n]$.

Algebra B is Koszul, with the Koszul dual algebra isomorphic to the cross-product of the exterior algebra on n generators with the group algebra of the symmetric group (recall that $\text{char}(\mathbb{k}) = 0$). Hence, B has finite Tor dimension and, in particular, $Z_0DH_n = \mathbb{k}[S_n]$ has Tor dimension n as a right B -module. B has Tor dimension 0 as a right Z_0DH_n -module, since $\mathbb{k}[S_n]$ is semisimple. We are in a position to invoke Quillen's theorem [38, Theorem 7, page 112], [5].

Theorem. *Let A be a ring equipped with an increasing filtration Z_kA , and such that Z_0A is regular. Suppose that $B = \text{gr}(A)$ has finite Tor dimension as a right Z_0A module and that Z_0A has finite Tor dimension as a right B module. Then the inclusion $Z_0A \subset A$ induces an isomorphism $K_i(Z_0A) \cong K_i(A)$.*

In our case $A = DH_n$. The regularity of Z_0DH_n is obvious due to it being semisimple (a regular ring is a noetherian ring such that every left module has finite projective dimension). Applying the theorem in $i = 0$ case we obtain

Proposition 8 *The inclusion $\mathbb{k}[S_n] \subset DH_n$ and the surjection $DH_n \rightarrow \mathbb{k}[S_n]$ induce mutually-inverse isomorphisms $K_0(\mathbb{k}[S_n]) \cong K_0(DH_n)$.*

Same argument shows that inclusion

$$\mathbb{k}[S_n] \subset DH_n \otimes \mathbb{k}[c_0, \dots, c_r]$$

induces an isomorphism of Grothendieck groups

$$K_0(\mathbb{k}[S_n]) \cong K_0(DH_n \otimes \mathbb{k}[c_0, \dots, c_r]).$$

Formation of Grothendieck groups commutes with taking direct limit of rings [40, Section 1.2], and Π is the limit of $\mathbb{k}[c_0, \dots, c_r]$ as $r \rightarrow \infty$. We obtain an isomorphism

$$K_0(\mathbb{k}[S_n]) \cong K_0(\text{End}(+^n))$$

proving Conjecture 1.1 when $m = 0$. \square

Induction and restriction functors for inclusions

$$\begin{aligned} \mathbb{k}[S_n] \otimes \mathbb{k}[S_m] &\subset \mathbb{k}[S_{n+m}] \\ DH_n \otimes DH_m &\subset DH_{n+m} \end{aligned}$$

induce “multiplication” and “comultiplication” maps on Grothendieck groups that turn

$$\bigoplus_{n \geq 0} K_0(\mathbb{k}[S_n]) \quad \text{and} \quad \bigoplus_{n \geq 0} K_0(DH_n)$$

into graded birings, see Geissinger [17] for the symmetric group, Zelevinsky [43] for semisimple generalizations, and Bergeron and Li [3], Khovanov and Lauda [23] for nonsemisimple ones. We write *birings* rather than *bialgebras*, since these K_0 groups are abelian groups rather than vector spaces over some field. Isomorphisms in the above proposition are compatible with multiplication and comultiplication, and induce a biring isomorphism

$$\bigoplus_{n \geq 0} K_0(DH_n) \cong \bigoplus_{n \geq 0} K_0(\mathbb{k}[S_n]).$$

The second biring can be canonically identified [17, 43] with an integral form Sym of the ring of symmetric functions in infinitely many variables.

What can we say about K_0 of $\text{End}(+^{n-m})$ when $n, m > 0$? Recall (end of Section 2.1) that $\text{End}(+^{n-m})$ contains 2-sided ideal $J_{n,m}$ spanned by basis elements of thickness less than $n + m$, which fits into short split exact sequence

$$0 \longrightarrow J_{n,m} \longrightarrow \text{End}(+^{n-m}) \longrightarrow DH_{n,m} \longrightarrow 0, \quad (43)$$

where

$$DH_{n,m} := DH_n \otimes DH_m \otimes \Pi$$

is the tensor product of two degenerate AHA with the polynomial algebra in infinitely many generators. The earlier argument via Quillen’s theorem shows that the inclusion

$$\mathbb{k}[S_n] \otimes \mathbb{k}[S_m] \longrightarrow DH_n \otimes DH_m \otimes \mathbb{k}[c_0, c_1, \dots, c_r]$$

induces an isomorphisms of K_0 -groups

$$K_0(\mathbb{k}[S_n] \otimes \mathbb{k}[S_m]) \cong K_0(DH_n \otimes DH_m \otimes \mathbb{k}[c_0, c_1, \dots, c_r]).$$

Passing to the limit $r \rightarrow \infty$, the inclusion

$$\mathbb{k}[S_n] \otimes \mathbb{k}[S_m] \longrightarrow DH_{n,m}$$

induces an isomorphism of Grothendieck groups

$$K_0(\mathbb{k}[S_n] \otimes \mathbb{k}[S_m]) \cong K_0(DH_{n,m}). \quad (44)$$

The split short exact sequence (43) gives rise to a split short exact sequence

$$0 \longrightarrow K_0(J_{n,m}) \longrightarrow K_0(\text{End}(+^n -^m)) \longrightarrow K_0(DH_{n,m}) \longrightarrow 0$$

(for the definition of the K -group of a 2-sided ideal see [40, 42]) and canonical decomposition

$$K_0(\text{End}(+^n -^m)) \cong K_0(DH_{n,m}) \oplus K_0(J_{n,m}).$$

Conjecture 1.1 is equivalent to the vanishing of $K_0(J_{n,m})$ for all n, m .

References

- [1] B. Bartlett, The geometry of unitary 2-representations of finite groups and their 2-characters, 2008, math.QA/0807.1329.
- [2] B. Bartlett, On unitary 2-representations of finite groups and topological quantum field theory, *PhD Thesis*, University of Sheffield, 2008, math.QA/0901.3975.
- [3] N. Bergeron and H. Li, Algebraic structures on Grothendieck groups of a tower of algebras, arXiv math/0612170.
- [4] G. Bergman, The diamond lemma for ring theory, *Adv. Math.* **29** (1978), 178–218.
- [5] J. Block, Cyclic homology of filtered algebras, *K-theory* **1** (1987), 515–518.
- [6] J. Brundan, Symmetric functions, parabolic category O , and the Springer fiber, *Duke Math. J.* **143**, no.1 (2008), 41–79, math.RT/0608235.
- [7] S. Cautis and A. Licata, Heisenberg categorification and Hilbert schemes, preprint 2010.
- [8] T. Ceccherini-Silberstein, F. Scarabotti, and F. Tolli, Representation theory of the symmetric groups. The Okounkov-Vershik approach, character formulas, and partition algebras, *Camb. Studies in Adv. Math.* **121**, 2010, Cambridge U Press.

- [9] I. Cherednik, Special bases of irreducible representations of degenerate affine Hecke algebra, *Func. Analysis and its Appl.* **20**, no. 1, (1986), 76–78.
- [10] I. Cherednik, A new interpretation of Gelfand-Tsetlin bases, *Duke Math. J.* **54** no.2, (1987) 563–577.
- [11] J. Cockett, J. Koslowski, and R. Seely, Introduction to linear bicategories, *Math. Structures Comput. Sci.* **10** no.2, (2002) 165–203.
- [12] P. Cvitanović, Group theory, Birdtracks, Lie’s and exceptional groups, 2008, Princeton U. Press, available online.
- [13] P. Diaconis and C. Greene, Applications of Murphy’s elements, *Stanford University Technical Report* (1989), no. 335.
- [14] V. Drinfeld, Degenerate affine Hecke algebras and Yangians, *Func. Analysis and its Appl.* **20** (1986), 62–64.
- [15] I. B. Frenkel, Private communication, circa 2003.
- [16] I. B. Frenkel, N. Jing, and W. Wang, Vertex representations via finite groups and the McKay correspondence, *Int. Math. Res. Notices* **4** (2000), 195–222, math.QA/9907166.
- [17] L. Geissinger, Hopf algebras of symmetric functions and class functions, in *Combinatoire et représentation du groupe symétrique, Lect. Notes in Math.*, 579, 168–181, Springer-Verlag, 1977.
- [18] I. Grojnowski, Instantons and affine algebras I: the Hilbert scheme and vertex operators, *Math. Res. Letters* **3** (1996), 275–291.
- [19] A. Guionnet, V. F. R. Jones, and D. Shlyakhtenko, Random matrices, free probability, planar algebras and subfactors, arxiv 0712.2904.
- [20] A. Hora, Jucys-Murphy element and walks on modified Young graph, *Banach Center Publ.* **73** (2006), 223–235.
- [21] M. Khovanov, A functor-valued invariant of tangles, *Alg. Geom. Top.* **2**, (2002) 665–741, math.QA/0103190.
- [22] M. Khovanov, Categorifications from planar diagrammatics, arXiv:1008.5084.
- [23] M. Khovanov and A. Lauda, A diagrammatic approach to categorification of quantum groups I, *Represent. Theory* **13** (2009), 309–347, arXiv:0803.4121.
- [24] M. Khovanov and A. Lauda, A diagrammatic approach to categorification of quantum groups III, *Quantum Topology*, **1**, Issue 1, (2010), 1–92, arXiv:0807.3250.

- [25] A. Kleshchev, Linear and projective representations of symmetric groups, *Cambridge Tracts in Mathematics 163*, 2005, Cambridge U. Press.
- [26] A. Lauda, Frobenius algebras and ambidextrous adjunctions, *Theory Appl. Categ.* **16**, (2006) 84–122, math.CT/0502550.
- [27] A. Lauda, A categorification of quantum $\mathfrak{sl}(2)$, to appear in *Adv. Math.*, math.QA/0803.3652.
- [28] G. Lusztig, Affine Hecke algebras and their graded version, *Journal of the AMS* **2**:3 (1989), 599–635.
- [29] S. MacLane, Categories for the working mathematician, Springer, 2nd edition, 1998.
- [30] A. Molev and G. Olshanski, Degenerate affine Hecke algebras and centralizer construction for the symmetric group, *J. Algebra* **237** (2001), 302–341, math.RT/0002165.
- [31] M. Müger, From subfactors to categories and topology I. Frobenius algebras in and Morita equivalence of tensor categories, *J. Pure Appl. Alg.* **180**, (2003) 81–157, math.CT/0111204.
- [32] H. Nakajima, Heisenberg algebra and Hilbert schemes of points on projective surfaces, *Ann. of Math.* **145**, (1997) 379–388.
- [33] H. Nakajima, Lectures on Hilbert schemes of points on surfaces, *University Lecture Series 18*, AMS, 1999.
- [34] A. Okounkov, Random matrices and random permutations, *Int. Math. Res. Notices.* (2000), no.20, 1043–1095, math.CO/990317.
- [35] A. Okounkov, A.Vershik, A new approach to representation theory of symmetric groups, *Selecta Math. (N.S.)* **2**, no. 4 (1996), 1–15.
- [36] A. Okounkov, A.Vershik, A new approach to the representation theory of the symmetric groups 2, *Zap. Nauchn. Sem. POMI* (in Russian), (2004) v.307, 57–98, math.RT/0503040.
- [37] G. Olshanski, Extensions of the algebra $U(g)$ for infinite-dimensional classical Lie algebras g and the Yangians $Y(gl(m))$, *Soviet Math. Dokl.* **36** (1988), 569–573.
- [38] D. Quillen, Higher algebraic K-theory **I**, *Lect. Notes in Math.* **341**, Springer, (1973), 85–147.
- [39] A. Rattan, Character polynomials and Lagrange inversion, *PhD Thesis*, University of Waterloo, 2005, available online.

- [40] J. Rosenberg, Algebraic K-theory and its applications, *Graduate Text in Mathematics* **147**, Springer, 1994.
- [41] P. Śniady, Asymptotics of characters of symmetric groups, genus expansion and free probability, *Discrete Math.* **306** no.77, (2006) 624–665, math.CO/0411647.
- [42] C. Weibel, The K-book: An introduction to algebraic K-theory, available online, <http://www.math.rutgers.edu/~weibel/Kbook.html>.
- [43] A. Zelevinsky, Representations of finite classical groups, *Lect. Notes in Math.* **869**, Springer, 1981.

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